The Configuration space integral for links and tangles in R^3

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Abstract

The perturbative expression of Chern-Simons theory for links in Euclidean 3-space is a linear combination of integrals on configuration spaces. This has successively been studied by Guadagnini, Martellini and Mintchev, Bar-Natan, Kontsevich, Bott and Taubes, D. Thurston, Altschuler and Freidel, Yang... We give a self-contained version of this study with a new choice of compactification, and we formulate a rationality result. Then we generalise the construction to braids and tangles by constructing the suitable limit of the configuration spaces when we stretch the braid or tangle to a vertical line. We show that the limit integral is multiplicative and we define the corresponding connection. We also define a braiding and an associator which determine our limit for tangles. The braiding is given as a function of the Bott and Taubes anomaly. If the anomaly was zero in degrees > 4, then our limit integral for links would be equal to the Kontsevich integral.

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1 Introduction

The perturbative expression of Chern-Simons theory for links in Euclidean 3-space in the Landau gauge, that we call "configuration space integral" (according to a suggestion of D.Bar-Natan), is a linear combination of integrals on configuration spaces. It has successively been studied by E. Guadagnini, M. Martellini and M. Mintchev [GMM], D.Bar-Natan [BN2], M. Kontsevich [Ko], Bott and Taubes [BT], Dylan Thurston [Th] (who explained in details the notion of degree of a map), Altschuler and Freidel [AF], Yang [Ya]... We give a self-contained version of this study with a new choice of compactification, and formulate a rationality result. Then we generalise the construction to braids and tangles by constructing the suitable limit of the configuration spaces when we stretch the braid or tangle to a vertical line. We show that the limit integral is multiplicative and we define the corresponding connection. We also define a braiding and an associator which determine our limit for tangles. The braiding is given as a function of the Bott and Taubes anomaly. If the anomaly was zero in degrees > 4, then this integral for links would be equal to the Kontsevich integral.[1]

This article is a new version of the two preprints [P1] and [P2], which contained all the main ideas except for the degree 3 anomaly and the Vassiliev universal property.

1.1 Definitions

Let M be a compact one-dimensional manifold with boundary. Let L denote an imbedding of M into $\mathbb{R}^3$. We say that L is a *link* if we moreover have the condition that the boundary of M is empty. And it is a *knot* when $M = S^1$.

A *diagram with support* M, is the data of :

1) A graph made of a finite set $V = U \cup T$ of vertices and a set E of edges which are pairs of elements of V, such that : the elements of U are univalent (belong to precisely one edge) and the elements of T are trivalent, and every connected component of this graph meets U.

2) An isotopy class σ of injections i of U into the interior of M.

This definition of a diagram differs from the usual one in that it excludes the double edges (—◯—) and the loops (∘–). This will be justified by the fact that the integral that we shall define make no sense on loops, and by Lemma 1.9 which will allow us to exclude diagrams with a double edge, among others.

For a diagram Γ, the *degree* of Γ is the number

$$\deg \Gamma = \frac{1}{2}\#V = \#E - \#T = \frac{1}{3}(\#E + \#U).$$

An *isomorphism* between two diagrams $\Gamma = (V, E, \sigma)$ and $\Gamma' = (V', E', \sigma')$, is a bijection from V to V' which transports E to E' and σ to σ'.

[1] There was a gap in the only proposed prepublished proof of the anomaly vanishing

Let $|\Gamma|$ denote the number of automorphisms of Γ.

Let $\mathcal{D}(M)$ denote the set of diagrams with support M up to isomorphism, and $\mathcal{D}_n(M)$ its subset made of the degree n diagrams, for all $n \in \mathbb{N}$.

$\mathcal{D}_0(M)$ has one element which is the empty diagram.

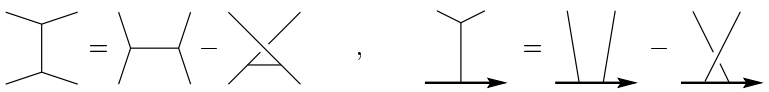
$\mathcal{D}_1(S^1)$ also has only one element, which will be denoted by θ, for it looks like the letter θ.

An *orientation* o of a diagram Γ is the data of an orientation of all its vertices, where the *orientation* of a trivalent vertex is a cyclic order on the set of the three edges it belongs to, and the *orientation* of a univalent vertex is a local orientation of M near its image by i.

Let $\mathcal{A}_n(M)$ denote the real vector space generated by the set of oriented diagrams of degree n, and quotiented by the following AS, IHX and STU relations. The image of an oriented diagram Γ in the space $\mathcal{A}_n$ is denoted by $[\Gamma]$.

The AS relation says that if Γ and Γ' differ by the orientation of one vertex, then $[\Gamma'] = -[\Gamma]$.

The IHX and STU relations are respectively defined by the following formulas.



These formulas relate diagrams which are identical outside one place, where they differ according to the figures. By convention for the figures, the orientation at each trivalent vertex is the given counterclockwise order of the edges, and the orientation of each univalent vertex is given by the drawn orientation of the bold line (which represents M locally).

Now let $\mathcal{A}(M)$ denote the space

$$\mathcal{A}(M) = \prod_{n \in \mathbb{N}} \mathcal{A}_n(M).$$

A *configuration* of a diagram $\Gamma \in \mathcal{D}(M)$ on L is a map from the set V of vertices of Γ to $\mathbb{R}^3$, which is injective on each edge, and which coincides on U with $L \circ i$ for some i in the class σ. The *configuration space* $C_\Gamma(L)$ (or simply C_Γ if L is fixed) of a diagram $\Gamma \in \mathcal{D}(M)$ on L is the set of these configurations.

C_Γ has a natural structure of a smooth manifold of dimension $\#U + 3\#T = 2\#E$. We shall canonically associate an orientation of C_Γ to the data of an orientation of Γ and an orientation of all edges of Γ, such that it is reversed when we change the orientation of one vertex or one edge (Subsection 4.2).

When orientations of all edges are chosen, we have the following canonical map Ψ from C_Γ to $(S^2)^E$: for a configuration $x \in C_\Gamma$ and $e = (v, v')$, $\Psi(x)_e$, is the direction of $x(v') - x(v)$.

Let ω denote the standard volume form on S^2 with total mass 1, and let

$$\Omega = \bigwedge^E \omega$$

denote the corresponding volume form on $(S^2)^E$.

Now the *configuration space integral* is $Z(L) \in \mathcal{A}(M)$:

$$Z(L) = \sum_{\Gamma \in \mathcal{D}(M)} \frac{I_L(\Gamma)}{|\Gamma|}[\Gamma]$$

where

$$I_L(\Gamma) = \int_{C_\Gamma(L)} \Psi^* \Omega.$$

We shall see that this integral $I_L(\Gamma)$ converges. The sign of the product $I_L(\Gamma)[\Gamma]$ will not depend on choices of orientations. Then $Z(L)$ also converges because it is a finite sum in each $\mathcal{A}_n(M)$.

By convention, the zero-degree part of $Z(L)$ is $1 = [\emptyset]$.

As an introduction to the study of $Z(L)$, let us first recall the properties of its degree one part when L is a link.

There are two types of degree one diagrams: the chords $c_{m,m'}$ between two different components m and m' of M, and the chords θ_m with both ends on the same component m of M. There are no IHX or STU relations between them, so the degree one part of Z is consists of the data of the integrals $I_L(c_{m,m'})$ and $I_L(\theta_m)$.

The space $C_{c_{m,m'}}(L)$ is identical to $m \times m'$, so, since M has no boundary, it is diffeomorphic to the torus $S^1 \times S^1$. Now, Ψ is a smooth map from $m \times m'$ to S^2, so the integral $I_L(c_{m,m'}) = \int_{c_{m,m'}} \Psi^* \omega$ is equal to the degree of the map Ψ: it takes values in $\mathbb{Z}$. Precisely, it is equal to the linking number of $L(m)$ and $L(m')$.

The space C_θ is diffeomorphic to the noncompact surface $S^1 \times]0,1[$. So the integral $I_L(\theta)$ is not the degree of a map. In fact it is known as the *Gauss integral*, usually considered in the case of a knot. It takes all real values on each isotopy class of knots. In the case of an almost planar knot, it approaches an integer which is the writhe or *self-linking number* of this knot (the number of crossings counted with signs).

In order to express some theorems about this expression $Z(L)$, we need first to recall some usual algebraic structures.

1.2 Algebraic structures on the spaces $\mathcal{A}$ of diagrams

For M and M' included in some M'' such that their interiors do not meet, one can consider the bilinear map from $\mathcal{A}(M) \times \mathcal{A}(M')$ to $\mathcal{A}(M'')$ defined by: for every oriented diagrams Γ with support M and Γ' with support M', $[\Gamma] \cdot [\Gamma'] = [\Gamma \sqcup \Gamma']$. This bilinear map is graded, that is, it sends $\mathcal{A}_n(M) \times \mathcal{A}_{n'}(M')$ to $\mathcal{A}_{n+n'}(M'')$.

Now, let $J = [0,1]$. It is well-known [BN1] that the bilinear product on $\mathcal{A}(J)$ defined by gluing the two copies of J preserving the orientation, defines a commutative algebra structure with unit $1 = [\emptyset]$, and that $\mathcal{A}(J)$ can be identified with $\mathcal{A}(S^1)$.

Similarly, for every M and every connected component m of M with an orientation (either with boundary or not), the insertion of J at some point of m preserving the orientation provides an $\mathcal{A}(J)$-module structure on $\mathcal{A}(M)$: it is well-known [BN1] that the result does not depend on the place of the insertion, modulo the AS, IHX and STU relations.

Algebra of diagrams on braids

We generalise the algebra $\mathcal{A}(J)$ in the following way: Let X be a nonempty finite set and $M = X \times J$. Then $\mathcal{A}(M)$ also has an algebra structure, defined by the natural gluing of two copies of M: in the product $[\Gamma_1] \cdot [\Gamma_2]$, the end $X \times \{1\}$ of Γ_1's support is glued to the end $X \times \{0\}$ of Γ_2's support. It is non-commutative if $\#X > 1$.

The duplication map. Let X and Y be two finite sets and $j \in X$. Let $M = X \times J$ and $M_j^Y = X_j^Y \times J$ where X_j^Y is obtained from X by replacing j by the elements of Y.

Definition 1.1 Define the *duplication map* $\Delta_j^Y : \mathcal{A}(M) \longrightarrow \mathcal{A}(M_j^Y)$ in the following way: for each diagram $\Gamma \in \mathcal{D}(M)$ with its map i from U to $X \times J$, let U_j denote $i^{-1}(\{j\} \times J)$. Then, let $\Delta_j^Y(\Gamma)$ be the sum of the $[\Gamma']$, where $\Gamma' \in \mathcal{D}(M_j^Y)$ is obtained from Γ by replacing i by the map which coincides with i on $U - U_j$ and with $f \times p_J \circ i$ on U_j, where f runs over the maps from U_j to Y.

It is easy to see that this Δ_j^Y linearly extends to $\mathcal{A}(M)$, and that it is a morphism of algebras. When Y has two elements, this map is simply denoted by Δ_j.

1.3 Results on the configuration space integral for links

The convergence of the integrals $I_L(\Gamma)$ comes from the following proposition which will be proved in Section 2.

Proposition 1.2 *There exists a compactification of C_Γ into a manifold with corners $\overline{C_\Gamma}$, on which the canonical map Ψ extends smoothly.*

Bott and Taubes first proved the convergence of the integrals by compactifying the configuration space C_Γ into a manifold with corners where $\Psi^*\Omega$ extends smoothly (their compactification was a little stronger than ours).

They restricted their interest to the case of knots, and they studied the variations of these integrals during isotopies, in order to check the invariance. So, they applied the Stokes theorem to express the variations of $I_L(\Gamma)$, in terms of integrals on the faces of the $\overline{C_\Gamma}$. They found equalities between the integrals at certain faces of different $\overline{C_\Gamma}$, where the diagrams Γ are the terms of an IHX or STU relation, and proved cancellation on the other faces except those of a special kind called "anomalous", which follow in some way the tangent map of L. The global effect of the contributions of the anomalous faces to the variations

of the integral can be expressed by a universal constant $\alpha \in \mathcal{A}$, the so-called *anomaly*.

Using many of their arguments, we shall prove the following proposition for links (when $\partial M = \emptyset$):

Proposition 1.3 *The variations of $Z(L)$ on an isotopy class of links are expressed by the formula*

$$Z(L) = Z^0(L) \prod_m \exp\left(I_L(\theta_m)\alpha^{(m)}\right)$$

where m runs over the connected components of M, and :
$Z^0(L) \in \mathcal{A}(M)$ is an isotopy invariant of L,
θ_m is the degree one diagram where both univalent vertices go to m,
α is a universal constant in $\mathcal{A}(J)$ called the anomaly*,*
$\alpha^{(m)}$ means α acting on $\mathcal{A}(M)$ on the component m.

But if the boundary of M was not empty, it would produce another boundary of the $\overline{C_\Gamma}$ and thus uncontrollable variations of Z.

Altschuler and Freidel [AF] have written this formula in the case of knots. They and Dylan Thurston [Th] also prove that

Proposition 1.4 *Z^0 is a universal Vassiliev invariant.*

The degree one part of α is $\frac{[\theta]}{2}$, and it is not difficult to see that all even degree parts of α cancel for symmetry reasons (Lemma 6.6).

We shall also prove in Section 7 that

Proposition 1.5 *The degree 3 part of the anomaly vanishes.*

But first we shall prove the following lemma in Subsection 2.1:

Notation 1.6 Let A be any nonempty subset of $V = U \cup T$. We denote

$$E_A = \{e \in E | e \subseteq A\}$$
$$E'_A = \{e \in E | \#(e \cap A) = 1\}$$

The cardinalities of these sets are related by the following formula which expresses the two ways of counting the half-edges of the vertices in A:

$$2\#E_A + \#E'_A = 3\#(A \cap T) + \#(A \cap U) \tag{1.7}$$

Definition 1.8 A diagram Γ is said to be *3-connected to U* (resp. 4-connected to U) if for any $A \subseteq T$ such that $\#A > 1$ we have $\#E'_A \geq 3$ (resp. ≥ 4).

Lemma 1.9 *If* $\operatorname{codim} \Psi(C_\Gamma) = 0$*, then Γ is 4-connected to U. And if* $\operatorname{codim} \Psi(C_\Gamma) = 0$ *or 1, then Γ is 3-connected to U.*

The idea of the proof of Proposition 1.3 is that for every n, the degree n part Z_n^0 of Z^0 can be seen as the degree of a certain map from a glued union of configuration spaces which are given rational multiples of the classes of their diagrams in $\mathcal{A}(M)$ as coefficients, to a certain manifold $(S^2)^N$ (where we can take for N the maximum number of edges of degree n diagrams).

This implies that Z_n^0 is a rational linear combination of the $[\Gamma]$ for the degree n diagrams Γ. More precisely we have the following proposition, which will be proved together with Proposition 1.3 in Section 6.

Notation 1.10 For a diagram Γ, let u_Γ be the number of univalent vertices of Γ, and e_Γ be its number of edges (we have $u_\Gamma + e_\Gamma = 3 \deg \Gamma$).

Let $k \in \mathbb{Z}$, $k \leq 2n$, and let $\mathcal{A}_n^k(M)$ be the quotient space of $\mathcal{A}_n(M)$ by the subspace generated by the diagrams Γ such that $u_\Gamma < k$, and that $\Psi(C_\Gamma(L))$ has codimension 0 or 1 for a generic L. We must suppose that $k \leq 2n$, else we would have $\mathcal{A}_n^k(M) = \{0\}$. Let Z_n^k be the image of Z_n in $\mathcal{A}_n^k(M)$. Lemma 1.9 implies that $\mathcal{A}_n^2(M) = \mathcal{A}_n(M)$, and that $\mathcal{A}_n^3(M) = \mathcal{A}_n(M)$ if $n > 1$.

Proposition 1.11 *We suppose L is such that for all m, $I_L(\theta_m)$ is an integer. Let $n \geq 1$, $k \in \mathbb{Z}$ and $N = 3n - k$. Then Z_n^k belongs to the integral lattice generated by the elements*

$$\frac{(N - e_\Gamma)!}{N!\, 2^{e_\Gamma}} [\Gamma]$$

in the space $\mathcal{A}_n^k(M)$, where Γ runs over the degree n diagrams such that $\Psi(C_\Gamma(L))$ has codimension 0 for a generic L.

Remark 1.12 Here are three results which will be proved in a next article:

- The above proposition can be improved by saying that Z_n belongs to the integral lattice generated by the elements

$$\frac{[\Gamma]}{e_\Gamma!\, 2^{e_\Gamma}}$$

where Γ runs over the diagrams Γ such that $\Psi(C_\Gamma)$ has codimension 0.

- For a generic L, $\Psi(C_\Gamma)$ has codimension 0 if and only if Γ is 4-connected to U.
- The degree 5 part of the anomaly cancels (this uses a Maple program).

The present rationality result can be compared to the one obtained for the Kontsevich integral: it was proved in [Le] that the degree n part of the Kontsevich integral is a linear combination of chord diagrams (or of all diagrams, which is the same), where the coefficients are multiples of

$$\frac{1}{(1!2! \cdots n!)^4 (n+1)}.$$

For large values of n, these denominators are greater than ours, but have smaller prime factors.

1.4 Extension to tangles and q-braids

Now, we shall extend the above constructions to the cases of tangles, q-braids and thus also q-tangles, and define a braiding and an associator. These notions were first constructed from the Kontsevich integral. For an overview of what has been done on this subject, see for example [BS].

To study the case of tangles, we shall change the regularity class of L for technical reasons: instead of smooth maps, we use semi-algebraic maps. See [BCR] for the definition and properties of semi-algebraic sets and maps.

Definition 1.13 We say that L is a *semi-algebraic tangle with altitudes* $[a,b]$ if:

- $L(M) \subseteq \mathbb{R}^2 \times [a,b]$ and $L(\partial M) \subseteq \mathbb{R}^2 \times \{a,b\}$;
- L is of class C^1 and semi-algebraic;
- no tangent vector at $L(\partial M)$ is horizontal;
- the altitude map is not constant on any open subset of M.

Notation 1.14 For $\lambda \in \mathbb{R}^+$, let h_λ be the following map:

$$\begin{aligned} h_\lambda : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x_1,x_2,x_3) &\longmapsto (\lambda x_1, \lambda x_2, x_3) \end{aligned}$$

Let $Z^l(L)$ be the limit of $Z(h_\lambda \circ L)$ when $\lambda \to 0$ (we shall prove that it exists).

This limit will correspond to the integrals $I^l_L(\Gamma)$ of the form $\Psi^*\Omega$ on the "limits $C^l_\Gamma(L)$ of the spaces $C_\Gamma(h_\lambda \circ L)$ when $\lambda \to 0$", which are semi-algebraic manifolds that we shall define. Their shape can have much more complicated singularities than manifolds with corners, but we won't need to study them precisely. This construction will be used to prove the multiplicativity of Z^l:

Proposition 1.15 *Suppose that M is a glued union $M = M' \cup M''$. Let L' and L'' be two semi-algebraic tangles with support M' and M'', and altitudes $[a,b]$ and $[b,c]$ respectively, so that their union $L = L' \cup L''$ is a semi-algebraic tangle with support M and altitudes $[a,c]$. Then we have*

$$Z^l(L) = Z^l(L') \cdot Z^l(L'').$$

Now, let us see the variations of Z^l during isotopies. Intuitively, the gluings of faces will also pass to the limit and, since the slices of different altitudes in L now contribute independently to the integral, the extra boundary of the configuration spaces due to the boundary of M will not move provided that L does not move at (∂M)'s altitudes (this will be formalised later). And the anomalous faces follow the tangent map of L, which is controlled at the limit by the horizontal tangent vectors of L. This gives the following Proposition 1.17 which will be proved rigourously in Subsection 9.4:

Definition 1.16 Let L and L' be two semi-algebraic tangles with support M which coincide on ∂M (we suppose the interval $[a,b]$ is fixed). A *semi-algebraic*

isotopy between L and L' is a continuous, semi-algebraic map ϕ from $[0,1] \times M$ to $\mathbb{R}^3$ such that $L = \phi(0,\cdot)$, $L' = \phi(1,\cdot)$, and $\forall\, t \in [0,1]$, $\phi(t,\cdot)$ is a semi-algebraic tangle which coincides with L on ∂M.

According to Theorem 8.8.5 of [BS] proved in [Na], any function of class C^2 on a compact subset of some $\mathbb{R}^n$ can be C^2-approximated by some polynomial. Thus, any smooth isotopy can be C^2-approximated by a semi-algebraic isotopy, because after replacing an isotopy by a semi-algebraic object, one can check that the perturbation on the boundary can be suppressed.

Proposition 1.17 *(i) If there is a semi-algebraic isotopy ϕ between L and L' such that the set of values of the horizontal tangent vectors to all tangles $\phi(t,\cdot)$ is finite (in the set of horizontal directions), then $Z^l(L) = Z^l(L')$.*

(ii) If an horizontal tangent vector to a component m of a tangle L moves by an angle of $\mu\pi$ radians, then the value of $Z^l(L)$ is multiplied by $\exp(-\mu\alpha^{(m)})$.

Now, consider braids: a braid is a tangle with no horizontal tangent vectors, and it can also be seen as a path in the space $\mathcal{C}_X$ of injections of a finite set X into the plane $\mathbb{R}^2$.

We shall define a connection over $\mathcal{C}_X$ whose holonomy will be equal to Z^l (here, we call *connection* a 1-form with values in a Lie algebra). Proposition 1.17 implies that this connection is flat.

Then, we shall compactify the space $\mathcal{C}_X$ into a manifold with corners $\overline{\mathcal{C}}_X$ and extend Z^l to paths in $\overline{\mathcal{C}}_X$, such that it remains continuous and constant on homotopy classes of paths. In other words, if we fix one end of the braid, Z^l defines a continuous map from the universal covering of $\overline{\mathcal{C}}_X$ to $\mathcal{A}(X \times J)$. We define q-braids as paths in $\overline{\mathcal{C}}_X$ whose ends belong to strata of $\overline{\mathcal{C}}_X$ with minimal dimension.

Proposition 1.18 (Duplication property of Z^l). *If a braid $\mathcal{B}$ with support M_j^Y is made of a braid B_1 with support M at a large scale and a braid $\mathcal{B}_2$ with support $Y \times J$ at an infinitely smaller scale (thus, a path in the boundary of $\mathcal{C}_{X_j^Y}$), then*

$$Z^l(\mathcal{B}) = Z^l(\mathcal{B}_2) \cdot \Delta_j^Y(Z^l(\mathcal{B}_1)).$$

Finally, we obtain an associator Φ in the natural way:

Definition 1.19 We define the elements $\Phi \in \mathcal{A}(\{1,2,3\})$ and $R \in \mathcal{A}(\{1,2\})$ by

$$\Phi = Z^l\left(\left|\!\diagup\!\right|\right) \qquad , \qquad R = Z^l\left(\cup\!\!\cap\right).$$

Here in the definition of Φ, the braid reaches the boundary of $\mathcal{C}_{\{1,2,3\}}$ at each end.

Proposition 1.20 *These R and Φ together verify the associator axioms [LM].*

Let us recall the associator axioms: the first 2 axioms, respectively called the pentagon and the hexagon axiom, can be seen as consequences of the isotopy invariance when we reconstruct the integral for suitable q-braids from R and Φ, according to the multiplicative and duplication properties of Z^l. The last two axioms are:

$$\Phi^{-1} = \Phi^{(321)}$$

$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$$

where $\Phi^{(321)}$ is Φ with the first and third strand exchanged, and ε_i is the projection which deletes the i-th strand and cancels all the diagrams with univalent vertices on it.

Then we shall prove the formulas which relate R to the anomaly:

Lemma 1.21

$$R \cdot 1_{\cap} = \exp(-\alpha)$$

and

Proposition 1.22

$$R = \exp \frac{\Delta(\alpha) - \alpha^{(1)} - \alpha^{(2)}}{2}.$$

Conclusion

As it will follow from Proposition 9.3 2), the integral at the thin slice containing an extremum of a tangle L is the inclusion of $Z^l\left(\cap\right)$.

The value of this element can be deduced from the value of Φ, with the help of the following remarks. First, we can see by symmetry around an horizontal axis that

$$Z^l\left(\cap\right) = Z^l\left(\cup\right).$$

Second, according to Proposition 1.17 we have

$$Z^l\left(\mathbb{N}\right) = Z^l\left(\,|\,\right)$$

Thus, the value of $Z^l(L)$ for all links L can be expressed in terms of R and Φ (and the limit values of the $I_L(\theta^{(m)})$) in the same way as the Kontsevich integral is deduced from the value $R = \exp(H/2)$ and the Drinfeld associator.

We deduce from Proposition 1.22 that the zero-anomaly result $\alpha = \frac{[\theta]}{2}$ would imply $R = \exp(H/2)$, where H is the diagram in $D^p_{\{1,2\}}$ with only one chord and the same vertical orientation at both univalent vertices. According to Theorem 8 of [LM], this would imply the equality of the configuration space integral with the Kontsevich integral.

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2 Compactification

Notation 2.1 If A is a finite set with at least two elements, let C^A denote the space of nonconstant maps from A to $\mathbb{R}^3$ quotiented by the translation-dilatations group (that is the group of translations and positive homotheties of $\mathbb{R}^3$).

Note that the space C^A is diffeomorphic to the sphere $S^{3\#A-4}$: choose an element $x \in A$ to be at the origin; then the images of the other elements of A are to be considered modulo the dilatations with center the origin. Thus it is compact.

2.1 Proof of Lemma 1.9

Let $A \subseteq T,\ \#A > 1$.

Consider the following commutative diagram:

$$\begin{array}{ccc} C_\Gamma & \xrightarrow{\Psi_\Gamma} & (S^2)^E \\ \downarrow & & \downarrow \\ C^A & \longrightarrow & (S^2)^{E_A} \end{array}$$

According to (1.7) we have

$$2\#E_A + \#E'_A = 3\#A \implies \dim C^A = \dim (S^2)^{E_A} + \#E'_A - 4.$$

So, if $\Psi(C_\Gamma)$ has codimension 0, then $\dim C^A \geq \dim (S^2)^{E_A}$ for all A, thus Γ is 4-connected to U. The second part of the lemma is deduced in the same way. □

2.2 The compactified space $\mathcal{H}$ of a configuration space $C(G)$ of a graph

Notation 2.2 Let G be a graph defined as a pair $G = (V, E)$ where V is the set of "vertices" and E is the set of "edges": V is a finite set and E is a set of pairs of elements of V. We suppose that $\#V \geq 2$ and that G is connected.

We define the configuration space $C(G)$ as the space of maps from V to $\mathbb{R}^3$ which map the two ends of every edge to two different points, modulo the dilatations and translations (it is a dense open subset of C^V).

By an abuse of notation, any subset A of V will also mean the graph with the set A of vertices and the set E_A of edges (the set of edges $e \in E$ such that $e \subseteq A$).

Let R be the set of subsets A of V such that $\#A \geq 2$ and A is connected.

Let $\mathcal{H}$ be the subset of $\prod_{A \in R} C^A$ made of the $x = (x_A)_{A \in R}$ such that $\forall A, B \in R,\ A \subseteq B \implies$ the restriction of x_B to A is either the constant map or the map x_A.

We define a canonical imbedding of $C(G)$ in $\mathcal{H}$ by restricting any $f \in C(G)$ to each of the $A \in R$: indeed, $\forall\, A \in R$, $\forall\, f \in C(G)$, f is not constant on A because A contains at least one edge, so the restriction of f to A is a well-defined element of C^A. Moreover, this is an imbedding because $V \in R$.

Note that $\mathcal{H}$ is a closed subset of the compact manifold $\prod_{A \in R} C^A$, so it is compact. In the next subsection, we shall prove that $\mathcal{H}$ is a manifold with corners, and that it is a compactification of $C(G)$.

Now we are going to describe the strata of $\mathcal{H}$. First, let us introduce this description intuitively:

An element x of a stratum of $\mathcal{H}$ will be a limit of elements of $C(G)$. This limit is first described by its projection to C^V; but some edges will collapse if x is not in the interior stratum $C(G)$. Group the collapsing edges into connected components A, zoom in on each such A and describe the relative positions of the vertices in A: it is an element of C^A. If some edges in A still collapse, do the same operations until all the relative positions of points in connected subdiagrams are described. The set of all sets A on which you have zoomed in form a subset S of R.

Notation 2.3 In all the following, S will denote any subset of R such that $V \in S$ and any two elements of S are either disjoint or included one into the other. Let $S' = S - \{V\}$.

For any $A \in R$, let G/A denote the graph obtained by identifying the elements of A into one vertex, and deleting the edges in E_A.

For all $A \in S$, let A/S denote the graph A quotiented by the greatest elements A_i of S strictly included in A (they are disjoint and any element of S strictly included in A is contained in one of them).

Identify the space $C(A/S)$ to a subset of C^A in the natural way (each $x \in C(A/S)$ is then constant on each of the sets A_i) and let

$$C(G,S) = \prod_{A \in S} C(A/S) \subseteq \prod_{A \in S} C^A.$$

$\forall\, A \subseteq V, A \neq \emptyset$, let $\bar{A}$ denote the smallest element of S containing A;
$\forall\, A \in S'$, let $\widehat{A}$ denote the smallest element of S that strictly contains A.

Let us first check that the last definitions make sense.

For any nonempty subset A of V, the smallest element of S containing A is well-defined because first $A \subseteq V \in S$, then two elements of S which contain A cannot be disjoint, so one of them is included in the other.

The definition of $\widehat{A}$ is justified in the same way. This gives S a tree structure.

Lemma 2.4 *1) For all S and all $x \in C(G,S)$, there is a unique $\bar{x} \in \mathcal{H}$ which is an extension of x to R. This defines an imbedding of $C(G,S)$ into $\mathcal{H}$.*

2) This $\bar{x}$ has the property that for all $A, B \in R$ with $A \subseteq B$, the restriction of $\bar{x}_B$ to A is constant if and only if $\bar{A} \neq \bar{B}$.

3) For all $x \in \mathcal{H}$, there is a unique S such that the restriction of x to S belongs to $C(G,S)$.

Proof:

1) First let us see the construction and unicity of $\bar{x}$.

Since $\bar{x}$ must belong to $\mathcal{H}$, for all $A \in R$, the restriction of $\bar{x}_{\bar{A}} = x_{\bar{A}}$ to A must be either constant or equal to $\bar{x}_A$. But it is not constant for the following reason:

By definition of $\bar{A}$, A is not reduced to one vertex in $\bar{A}/S$. Furthermore, since A is connected, it has at least one common edge with $\bar{A}/S$. Then, since $x_{\bar{A}} \in C(\bar{A}/S)$, it cannot be constant on this edge.

Thus, for all $A \in R$, the value of $\bar{x}_A$ is determined by $x_{\bar{A}}$. This proves the unicity of $\bar{x}$.

As for the existence of $\bar{x}$, the fact that this $\bar{x}$ constructed above is actually an element of $\mathcal{H}$, will be a consequence of 2).

Now, the fact that this defines an imbedding of $C(G,S)$ into $\mathcal{H}$ is easy.

2) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$, so if $\bar{A} \neq \bar{B}$ then the restriction of $x_{\bar{B}}$ to $\bar{A}$ is constant, so the restriction of $\bar{x}_B$ to A is also constant.

If $\bar{A} = \bar{B}$, then x_A and x_B are both restrictions of $x_{\bar{A}}$, so x_A is the restriction of x_B to A.

3) The existence and unicity of such an S come from the following construction of S as a function of an $x \in \mathcal{H}$.

The elements of S can be enumerated recursively, starting with its first element $A = V$. For any $A \in S$, consider the partition of A defined by x_A (i.e. the set of preimages of the singletons). Then, the greatest elements of S strictly included in A must be precisely the connected components, with cardinality greater than one, of the sets in this partition. □

2.3 Description of the corners of $\mathcal{H}$

Now we identify $C(G,S)$ with a subset of $\mathcal{H}$ thanks to Lemma 2.4. A direct calculation shows that $\dim C(G,S) = \dim C(G) - \#S'$. We are going to see the following

Proposition 2.5 *$C(G,S)$ has a neighbourhood[2] in $\mathcal{H}$ which is diffeomorphic to a neighbourhood of $C(G,S) \times \{0\}^{S'}$ in $C(G,S) \times [0,+\infty[^{S'}$. Thus, $\mathcal{H}$ is a manifold with corners.*

Intuitively, in the family of parameters $(u_A)_{A \in S'}$, each u_A will measure the relative size of A and $\widehat{A}$.

This diffeomorphism is not completely canonical: to define it we have to choose an identification of $C(A/S)$ with a smooth section of representatives in $(R_A)^{A/S}$, where R_A is just a copy of $\mathbb{R}^3$ marked with the label A, for each $A \in S$. For instance, suppose that V is totally ordered and that for all $A \in R$ we fix the smallest vertex in A at the origin of R_A; as for the dilatations, just fix any choice of normalisation.

[2] A neighbourhood of a subset P of a topological space is a neighbourhood of every point of P; it does not necessarily contain the closure of P.

1) *Construction of a smooth map from a neighbourhood of* $C(G,S) \times \{0\}^{S'}$ *in* $C(G,S) \times [0,+\infty[^{S'}$ *to a neighbourhood of* $C(G,S)$ *in* $\mathcal{H}$.

Take a family $u = (u_A)_{A \in S'}$ of positive real numbers and a family $f \in C(G,S)$, that is, $f = (f_A)_{A \in S}$ and $f_A \in R_A^{A/S}$. Then, define the net (ϕ_A) of correspondences from each space R_A for $A \in S'$ to the space $R_{\widehat{A}}$ by

$$\begin{aligned} \phi_A : R_A &\longrightarrow R_{\widehat{A}} \\ x &\longmapsto f_{\widehat{A}}(A) + u_A x \end{aligned}$$

We construct an element g of $\mathcal{H}$ in the following way:

For all $B \in R$, we define the projection of g in C^B as a map from B to $R_{\bar{B}}$: first take for all $v \in B$ its image by $f_{\overline{\{v\}}}$ in $R_{\overline{\{v\}}}$, then compose all the maps ϕ_A above for the $A \in S$ such that $\overline{\{v\}} \subseteq A \subsetneq \bar{B}$ to obtain an element of $R_{\bar{B}}$.

This map is not constant on B for small enough values of u because for $u = 0$ it is equal to $f_{\bar{B}}$ which is not constant on B, so it gives an element of C^B.

This provides a well-defined smooth map from a neighbourhood $\mathcal{U}$ of $C(G,S) \times \{0\}^{S'}$ in $C(G,S) \times [0,+\infty[^{S'}$, to a subset of $\mathcal{H}$, and its restriction to $C(G,S) \times \{0\}$ is the imbedding defined in Lemma 2.4.

It maps the interior of $\mathcal{U}$ to the stratum $C(G)$ of $\mathcal{H}$, because when $\forall\, A \in S'$, $u_A > 0$, these operations are equivalent to first constructing the projection of g in C^V, then restricting it to each element of R, because all the maps ϕ_A are dilatations-translations of $\mathbb{R}^3$. So it maps $\mathcal{U}$ to $\mathcal{H}$.

Remark 2.6 This diffeomorphism maps any $(f,u) \in \mathcal{U}$ into the stratum $C(G,S_u)$ where $S_u = \{A \in S | A = V \text{ or } u_A = 0\}$.

2) *Proof that the map above has a smooth inverse:*

We have to check that the u_A ($A \in S'$) and the f_A ($A \in S$) can be expressed as smooth functions of a g in a suitable neighbourhood of $C(G,S)$ in $\mathcal{H}$.

$f_A \in C(A/S)$ is the modification of g_A obtained by mapping each $A' \in S$ such that $\widehat{A'} = A$ to $g_A(a')$ where a' is the smallest element of A'. This f_A maps the two ends of every edge of A/S to two different points because it is in a suitable neighbourhood of $C(G,S)$ in $\mathcal{H}$.

u_A is the suitably defined size of A in $g_{\widehat{A}}$. □

The description of $\mathcal{H}$ is now finished and we can proceed with the compactification of C_Γ.

3 The compactified space $\overline{C_\Gamma}$ of C_Γ

3.1 Construction of $\overline{C_\Gamma}$

Definition of $\overline{C_\Gamma}$

Starting with a diagram $\Gamma \in \mathcal{D}(M)$, define the graph G with the same set V of vertices as Γ, and define the set of edges of G as being the set of edges of Γ, plus all pairs of univalent vertices.

First, canonically imbed the space C_Γ into the space

$$\mathcal{H}' = \mathcal{H} \times M^U$$

and let the compactified space $\overline{C_\Gamma}$ of C_Γ be its closure in $\mathcal{H}'$.

Now we are going to describe the corners of this space. For convenience, we suppose that M is oriented and $\partial M = \emptyset$. The case when $\partial M \neq \emptyset$ will be seen at the end of this subsection.

List of strata

A stratum of C'_Γ is labelled by the following data of $(S, P, (\leq)_P)$:

• A set $S \subseteq R$ as above (Notation 2.3)

• A partition P of U, which satisfies one of the relations:

(i) $P = \{U\}$

(ii) $P =$ the partition of U induced by $\bar{U}/S$.

• A total order on each element of P.

such that: for any $A \in P$ and $B \in S$, $A \cap B$ with its total order is a set of consecutive elements for σ (thus, A is mapped to a single connected component of M, and there is only one possibility for this total order if A is not the whole preimage of a connected component of M).

Remark 3.1 U is not reduced to a point in $\bar{U}/S$ because U is connected in the graph G, so it belongs to R. Thus, the above (i) and (ii) exclude each other. The case (i) is possible only if U is mapped to a single connected component of M.

Description of strata

The stratum with label $(S, P, (\leq)_P)$ is the set of elements $(f, f') \in \mathcal{H} \times M^U$ such that:

• f' belongs to the closure of σ and maps two univalent vertices to the same element of M if and only if they belong to the same element of P.

• $f \in C(G, S)$

• Each ordered pair (x, y) of elements of U which belong to the same element U_1 of P and are consecutive for $\leq_{U_1}$ is mapped by $f_{\{x,y\}}$ to the tangent vector to L at their common image. Or equivalently, $\forall\, A \in S$, if $A \cap U$ is contained in one element U_1 of P, then f_A maps $A \cap U$ to the straight line l_s with direction the tangent vector s to $L \circ f'(U_1)$, preserving the order in the large sense.

• In case (ii), $f_{\bar{U}}$ must coincide on U with $L \circ f'$. Thus, we identify $R_{\bar{U}}$ to the image space $\mathbb{R}^3$ of L, because $L \circ f'$ and $f_{\bar{U}}$ are not constant on U.

Note that in case (ii), the vertices in $V - \bar{U}$ are those which escape to infinity, or for which any path to a univalent vertex passes through trivalent vertices which escape to infinity (in case (i) there can be an undetermination).

We suppose that the univalent vertices are smaller so that they have priority to be at the origin of the spaces R_A.

The neighbourhood of a stratum

This is again a corner, with a family of independent positive parameters which are the same as before (indexed by S'), plus one more in the case (i), which will be denoted by u_0.

To be quick, we shall only define the elements of C_Γ which correspond to a family of strictly positive coordinates $(u_A)_{A\in S'}$, and possibly u_0.

This will be made in two steps: first an approximate definition, then a correction.

In the first step, we start with an element (f, f') of the considered stratum with $f \in C(G, S)$, and a family (u_A) of small enough strictly positive real numbers. The construction of Subsection 2.3 gives an identification of all the spaces R_A together (because $u_A > 0$ for all A), and thus an element of $C(G)$. There remains to identify one of the spaces (namely $R_{\bar{U}}$) to the image space $\mathbb{R}^3$ of L. In case (ii), this identification is already done; whereas in case (i), it will be

$$\begin{aligned} R_{\bar{U}} &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto L \circ f'(U) + u_0 x. \end{aligned}$$

In the second step, we have to choose for each Q in the image of $L \circ f'$ a diffeomorphism φ_Q between two neighbourhoods of Q which verifies the two conditions: first, it approximates the identity near Q up to the first order; second, it maps the tangent line to the curve $L(M)$ at Q, to the curve $L(M)$ itself. This system of diffeomorphisms must depend smoothly on f.

Then for every Q, correct the map from V to $\mathbb{R}^3$ of the first step by composing it with the map φ_Q for the vertices in ${f_{\bar{U}}}^{-1}(Q)$.

It can easily be seen that the resulting map extends as a smooth map on a neighbourhood of the stratum.

Case when $\partial M \neq \emptyset$

To the data $(S, P, (\leq)_P)$ we must add the datum of the set $f'(U) \cap \partial M$. As for the parametrisation of the corners, each $x \in f'(U) \cap \partial M$ gives the parameter $\mathrm{dist}(x, f_1'(U))$ for the (f_1, f_1') in the neighbourhood of the stratum. □

Proof of Proposition 1.2:

Now that we have compactified C_Γ into a manifold with corners, we just have to check that Ψ is smooth on $\overline{C_\Gamma}$. But Ψ is just the map defined by the canonical projections of $\mathcal{H}$ on the $C^e \approx S^2$ for $e \in E$, since $E \subseteq R$. □

3.2 List of faces

The codimension 1 strata of $\overline{C_\Gamma}$ as constructed in the previous subsection can be classified into six types. In the first five types we have $f'(U) \cap \partial M = \emptyset$.

(a) In case (i) there is the coordinate u_0, so S must be reduced to $\{V\}$.

In case (ii) with $S = \{V, A\}$, let us distinguish four types:

(b) $U \subseteq A$ (this corresponds to the case when the vertices not in A go to infinity),

(c) $\#A = 2$ and $E'_A \neq \emptyset$; this is divided into two subtypes:

(c1) $A \subseteq T$ (thus $A \in E$),
(c2) $A \not\subseteq T$ (thus, either $A \subseteq U$ and $A \notin E$, or $A \not\subseteq U$ and $A \in E$).
(d) $\#A > 2$, $U \not\subseteq A$, and $E'_A \neq \emptyset$,
(a') $E'_A = \emptyset$ (thus, $A \cap U \neq \emptyset$ and $U \not\subseteq A$).
Finally,
(e) Case (ii) with $S = \{V\}$ and $\#(f'(U) \cap \partial M) = 1$.

Definition 3.2 The faces of types (b), (c) and (d) will be called *ordinary faces.*
The faces of types (a) and (a') will be called *anomalous faces*[3].
The faces of type (e) will be called *extra faces.*
We say that a face F is *degenerated* if $\operatorname{codim} \Psi(F) > 1$.

Notation 3.3 For $A \in R$ with $U \not\subseteq A$, let $F(\Gamma, A)$ denote the face of C_Γ defined by $S = \{V, A\}$.

In the next section we shall prove the following proposition about some types of faces which are degenerated. Then, the degenerated faces can be eliminated.

Proposition 3.4 *The only faces of $\overline{C_\Gamma}$ which can be non-degenerated are:*
The faces of type (a) with Γ connected;
the faces $F(\Gamma, A)$ of type (a') with A connected;
the faces of types (c) and (e);
the faces $F(\Gamma, A)$ of type (d) such that each edge in E'_A meets at least two edges in E_A.

3.3 Proofs that some faces are degenerated

Remark 3.5 Suppose there exists a subset A of V such that $E'_A = \emptyset$. Then consider the two subdiagrams Γ_1 and Γ_2 whose sets of vertices are A and $V - A$ respectively. Then there is a canonical diffeomorphism from C_Γ to an open subset of $C_{\Gamma_1} \times C_{\Gamma_2}$, which is delimited by a finite union of subspaces.

This gives a diffeomorphism between a type (a') face, and an open subset of the product of a type (a) face of C_{Γ_1} with the space C_{Γ_2}. So it shows that the (a) and (a') types are fundamentally the same.
It also has the following consequence:

Lemma 3.6 *A type (a) face with Γ not connected, or a type (a') face with A not connected, is degenerated.*

Proof: Now, for the type (a), if Γ is not connected, we see that the fiber over an $f(U) \in S^1$ is mapped to the product of fibers corresponding to the different subdiagrams, and the expected dimension of the image decreases by two (because of the relative dilatations and translations of the different components).
This concludes for the type (a), and the generalisation to (a') is immediate. □

[3] They are the "anomalous faces" of Bott and Taubes.

Lemma 3.7 *Any type (b) face is degenerated (and the codimension of its image is greater than 3).*

Proof :

In the diagram $\Gamma/\bar{U}$, all vertices are at least trivalent because Γ is 3-connected to U. Moreover, its configuration space may be quotiented by dilatations and translations. Thus, its image by Ψ is contained in a codimension 4 subset of $(S^2)^E$ (which does not depend on L). □

Lemma 3.8 *Any type (d) face such that the vertex v in A of some edge in E'_A is either in U or not bivalent in A, is degenerated.*

Indeed, in such a face, each preimage of a point by Ψ is at least one-dimensional because in C^A, this vertex v can move freely in the tangent direction of $L(A)$ if $v \in U$, and in the direction of its edge in E_A otherwise. This movement is not trivial because $\#A > 2$. □

This ends the proof of Proposition 3.4.

4 Labelled diagrams and orientations

4.1 Labelled diagrams

To formulate the degree n part of Z^0 in terms of the degree of the map Ψ from the glued configuration spaces to a product of spheres, we shall need to fix this product of spheres. Thus, all our edges will be labelled and oriented. We shall call that "labellings" of diagrams. We are going to reformulate the definition of the diagrams so that they will be intrinsically labelled.

Let us fix the degree n of the diagrams. Their number of edges varies under STU but we must fix the number N of their labels. So, we shall only work with the diagrams Γ which have a number of edges $e_\Gamma \leq N$, and thus a number of univalent vertices $u_\Gamma \geq k = 3n - N$, and there may be some unused labels which will be called "absent edges".

Notation 4.1 If A is a set of sets, let $\cup A$ denote the union of elements of A.

Let $n \geq 1$, $k \leq 2n$ be fixed integers and $N = 3n - k$.

Let $E = \{e_1, \cdots, e_N\}$ be the set of labelled "edges" $e_i = \{2i-1, 2i\}$. The set $E_2 = \cup E = \{1, \cdots, 2N\}$ is called the set of half-edges.

Definition 4.2 Let $D_{n,k}(M)$ denote the set of 4-tuples $\Gamma = (U, T, E^v, \sigma)$, called *labelled diagrams*, which respect the following conditions:

- $E^v \subseteq E$ is the set of "visible edges", it labels the edges of the diagram. We let $E^a = E - E^v$ denote the set of "absent edges".
- The set $V = U \cup T$ of all vertices of Γ is a partition of $\cup E^v \subseteq E_2$.
- U is the set of univalent vertices: it is a set of singletons, and σ is an isotopy class of injections of U into the interior of M.
- T is the set of trivalent vertices, which are subsets of E_2 with three elements belonging to three different edges.

- $\#V = 2n$ (which is equivalent to $u_\Gamma = \#E^a + k$).
- Γ is 3-connected to U (see Definition 1.8).

The identity is the only isomorphism of diagrams which preserves the labelling, so we will take the same notion of an isomorphism as before, which refers to the underlying unlabelled diagrams:

Definition 4.3 An *isomorphism* between two labelled diagrams $\Gamma = (U_\Gamma, T_\Gamma, E^v_\Gamma, \sigma_\Gamma)$ and $\Gamma' = (U_{\Gamma'}, T_{\Gamma'}, E^v_{\Gamma'}, \sigma_{\Gamma'})$ (or *change of labelling*) is a bijection between $\cup E^v_\Gamma$ and $\cup E^v_{\Gamma'}$ which carries E^v_Γ to $E^v_{\Gamma'}$, T_Γ to $T_{\Gamma'}$, U_Γ to $U_{\Gamma'}$ and σ_Γ to $\sigma_{\Gamma'}$.

A configuration for a labelled diagram induces a map from $\cup E^v$ to $\mathbb{R}^3$, so the canonical map Ψ can be written:

$$\begin{aligned} \Psi_\Gamma : C_\Gamma &\longrightarrow (S^2)^{E^v} \\ f &\longmapsto \left(\frac{f(2i) - f(2i-1)}{|f(2i) - f(2i-1)|} \right)_{e_i \in E^v}. \end{aligned}$$

Now let Ψ' be the product of Ψ by the identity on $(S^2)^{E^a}$. So it is a map from the space $C'_\Gamma = C_\Gamma \times (S^2)^{E^a}$ to $(S^2)^E$.

4.2 Orientation of the configuration spaces

We shall now define an orientation of the space C'_Γ depending on the orientation o of Γ. It will be done by labelling all components of a natural local coordinate system of C'_Γ by the elements of E_2. These components are:

- One coordinate in M for each univalent vertex, respecting the local orientation of M given by the orientation of this vertex.
- The three standard components in $\mathbb{R}^3$ for each trivalent vertex.
- And for each absent edge we have the two components of a local coordinate system in S^2 with the direct orientation.

The coordinate in M for a univalent vertex $\{x\}$ is labelled by x. The two components for each $e_i \in E^a$ are respectively labelled by $2i-1$ and $2i$.

As for the trivalent vertices, let us represent o by a family of bijections $(b_t)_{t \in T}$ from each trivalent vertex t to the set $\{1,2,3\}$. Then, for all $x \in t$, give the label x to the $b_t(x)$-th component of $f(t)$ in $\mathbb{R}^3$.

Thus we have an orientation of C'_Γ, which only depends on the orientation o and the labelling of Γ. It is reversed when we change the orientation of one vertex or one edge; and the isomorphism induced by a permutation of E, which preserves the orientations of edges and vertices, also preserves the orientation of C'_Γ. So, this construction is equivalent to first orienting C_Γ by the same method with edges e_i $(i = 1, \cdots, \#E^v)$, then taking the direct product with $(S^2)^{E^a}$.

For a labelled oriented diagram (Γ, o), we can now define the integral

$$I_L(\Gamma, o) = \int_{C'_\Gamma} \Psi'^*_\Gamma \Omega = \int_{C_\Gamma} \Psi^*_\Gamma \Omega_{E^v}$$

where Ω is the standard volume form on $(S^2)^E$ with total mass 1, and Ω_{E^v} is the one of $(S^2)^{E^v}$.

Note that we have $I_L(\Gamma, -o) = -I_L(\Gamma, o)$ and that $I_L(\Gamma, o)$ is independent of the labelling of Γ: exchanging two edges preserves both orientations of C'_Γ and $(S^2)^E$, and reversing one edge changes both orientations (in other words, it changes both the orientation of C'_Γ and the sign of the form ${\Psi'_\Gamma}^*\Omega$), because the antipodal map reverses the orientation of S^2.

So, the product $I_L(\Gamma)[\Gamma]$ is well-defined and does not depend on o.

In all the following, we fix an arbitrary choice of orientations on the diagrams in $D_{n,k}(M)$, with no consequence on the final results.

4.3 Interpretation of Z in degree theory

Let β be the map from $D_{n,k}(M)$ to $\mathcal{A}^k_n(M)$ defined by:

$$\beta(\Gamma) = \text{projection of } \frac{(N - e_\Gamma)!}{N!\, 2^{e_\Gamma}}[\Gamma] \text{ in } \mathcal{A}^k_n(M). \tag{4.4}$$

It has the property that

$$\forall\, \Gamma \in D_{n,k}(M), \qquad \sum_{\Gamma' \in D_{n,k}(M), \Gamma' \sim \Gamma} \beta(\Gamma') = \frac{[\Gamma]}{|\Gamma|} \tag{4.5}$$

Let C' be the following formal linear combination of spaces

$$C' = \sum_{\Gamma \in D_{n,k}(M)} \beta(\Gamma) C'_\Gamma$$

Here, $-C'_\Gamma$ stands for C'_Γ with the opposite orientation. Let ξ be the almost everywhere defined map with rational values in $\mathcal{A}^k_n(M)$:

$$\begin{aligned} \xi : (S^2)^E &\longrightarrow \mathcal{A}^k_n(M) \\ x &\longmapsto \text{ algebraic number of } \Psi^{-1}(x) \text{ in } C' \end{aligned}$$

According to the degree theory, this map can be extended into a locally constant map defined outside the union of the $\Psi(\partial C'_\Gamma)$ for $\Gamma \in D_{n,k}(M)$. When a map is defined on a manifold without boundary, the number of preimages counted with signs is constant and called the *degree* of the map. In this article, we shall generalise the notion of "manifold without boundary", by defining a notion of gluing of faces in the next section, such that only the faces which are not glued are relevant in the notion of a boundary. Now, we have

$$Z^k_n = \int_{(S^2)^E} \xi \Omega.$$

So, Z^k_n is generally not rational because the map ξ is not constant on $(S^2)^E$. The aim of the following sections will be to prove Propositions 1.3 and 1.11 in

the following way: we shall introduce other configuration spaces to complete C' so that the corresponding ξ becomes constant.

Then, each of the expressions Z^0 of Proposition 1.3 and Z_n^k of Proposition 1.11 will be equal to the degree of the map Ψ defined on such a linear combination of configuration spaces which extends C'.

Remark 4.6 We could generalise these constructions to other maps β from $D_{n,k}(M)$ to any $\mathbb{Z}$-module. This might be used: either to search for torsion invariants by means of a torsion $\mathbb{Z}$-module; or to improve the rationality results by means of another map β which satisfies Formula (4.5) and optimised gluing conditions. We develop this in the Appendix. However, the improvement of Remark 1.12 is not based on this method.

4.4 Yang's point of view

Yang [Ya] introduced another interpretation of Z in terms of degree theory, which is sometimes useful to avoid some computations of coefficients. Here we must keep Formula (4.4) for β.

The idea is to quotient the space $(S^2)^E$ by the group $\mathcal{G}$ of permutations and reversals of the edges, to obtain the space $B_E = (\mathbb{RP}^2)^E/\Sigma_E$ where Σ_E is the permutation group of E. The group $\mathcal{G}$ also naturally acts on C', so that this action commutes with Ψ. Thus, the map ξ is invariant under the action of $\mathcal{G}$ (for the same reason as the fact that $I_L(\Gamma)[\Gamma]$ did not depend on orientations) and induces a map defined on B_E.

To understand the quotient space $C'/\mathcal{G}$, first note that the group of automorphisms of a diagram Γ freely acts on C_Γ. Let $C_\Gamma/|\Gamma|$ denote the quotient space of C_Γ by this action. It is almost everywhere identical to the space of all "configured diagrams" isomorphic to Γ, that is, unlabelled diagrams where the set V of vertices is a subset of the space $\mathbb{R}^3$ and the univalent vertices belong to the image of L. Then $C'/\mathcal{G}$ is topologically the union for all isomorphism classes of degree n diagrams Γ with $u_\Gamma \geq k$, of the product $C_\Gamma/|\Gamma| \times B_{E^a}$.

Now, the expression Z_n^k can be interpreted as an integral on these quotients, but not of a $2N$-form because some $I_L(\Gamma)$ can contribute even when the quotient space $C_\Gamma/|\Gamma|$ is not orientable (the simplest example is $\Gamma = \theta$). So we must replace the form $\Psi^*\Omega$ by the density $\Psi^*\Omega \cdot (\text{local orientation of } C_\Gamma)$.

Let us release the condition $u_\Gamma \geq k$ (or equivalently, take $k = 2$); replace C' by the space C constructed the same way as $C'/\mathcal{G}$ but without the absent edges (it is the union of the $C_\Gamma/|\Gamma|$ for degree n diagrams Γ up to isomorphism). Thus it is made of parts with different dimensions.

So, we can say that

$$Z_n = \int_C (\Psi^*\Omega \cdot (\text{orientation}) \cdot [\Gamma])_{local} \cdot$$

The whole integral Z itself is defined in the same way with diagrams of all degrees.

Note that the rational coefficient disappears in the above integral formula because of the volumes of B_E and B_{E^a}. The advantage of this presentation is that it generally trivialises the problem of handling of the rational coefficients of Formula (4.4). The bad point is that it gives no means to handle the sign of the density involved in the space C.

5 Ordinary gluings

5.1 Definition of a gluing

In this subsection, we explain in all generality (for any map β and any notion of a configuration space) the notion of a gluing of a set of faces:

Each face F of a configuration space C'_Γ contributes to a variation of $\beta(\Gamma)$ of the map ξ while going across $\Psi(F)$. Here, $\Psi(F)$ is counted with possible multiplicities. We are going to define *gluings* for certain sets of faces, that is, objects whose existence ensure that the contributions of these faces to the discontinuity of ξ at their images cancel. Note that the degenerated faces do not contribute to the discontinuity of ξ, thus no gluing is necessary for them.

Definition 5.1 A Ψ-*isomorphism* between two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ with maps Ψ from each of them to $(S^2)^N$, is a diffeomorphism φ from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that $\Psi \circ \varphi = \Psi$ on $\mathcal{M}_1$. A Ψ-isomorphism is said to be *positive* if it preserves their orientations.

Definition 5.2 A *self-gluing* of a face is a negative Ψ-automorphism of this face.

The existence of a self-gluing of a face clearly implies that this face does not contribute to the discontinuity of ξ. But self-gluings are quite exceptional and we shall use them only to eliminate some pathological cases of the other gluings.

In this section, we define gluings in the following way (but for the anomalous faces in the next section, it will be necessary to cut some faces into pieces).

Definition 5.3 A *gluing* of a set of faces $F(\Gamma, A)$ with coefficients $\beta(\Gamma)$ is a partition of this set such that all the faces in each class are Ψ-isomorphic and verify the *gluing condition*

$$\sum_{(\Gamma,A) \text{ in the class}} \pm\beta(\Gamma) = 0$$

where the signs in front of $\beta(\Gamma)$ and $\beta(\Gamma')$ for two given pairs (Γ, A) and (Γ', A') will be equal iff each Ψ-isomorphism between $F(\Gamma, A)$ and $F(\Gamma', A')$ preserves their orientations inherited from the orientations of the spaces C'_Γ and $C'_{\Gamma'}$.

Note that if there are both a positive and a negative Ψ-isomorphism between two faces, then each face in the class is glued to itself, so the condition may be deleted. Otherwise, the condition for the signs in the above formula is consistent.

5.2 Gluing results for ordinary faces

The aim of this section is to prove the following proposition.

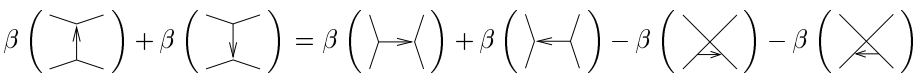
Proposition 5.4 *There exists a gluing of the set of ordinary, non-degenerated faces of the linear combination of spaces*

$$C' = \sum_{\Gamma \in D_{n,k}(M)} \beta(\Gamma) C'_\Gamma.$$

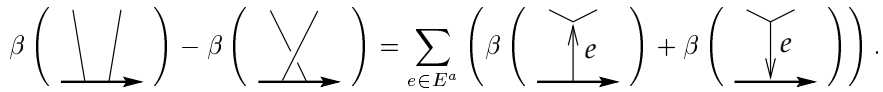
where β is the map from $D_{n,k}(M)$ to $\mathcal{A}_n^k(M)$ defined by Formula (4.4).

This is the consequence of the three following lemmas, which will be proved in Subsection 5.4. To formulate them, the map β which was defined on the set of diagrams $D_{n,k}(M)$ with chosen orientations, must be seen as extended to diagrams of all orientations according to the AS relation.

Lemma 5.5 *The gluings of the type (c1) faces are ensured by the following $\mathcal{RIHX}$ relation between labelled diagrams that are identical outside the drawn part, and all belong to $D_{n,k}(M)$:*



Lemma 5.6 *The gluings of the type (c2) faces are ensured by the following $\mathcal{RSTU}$ relation between labelled diagrams that are identical outside the drawn part and such that all three unlabelled diagrams Γ of the corresponding STU relation verify* $\operatorname{codim} \Psi(C_\Gamma) = 0$ *or* 1*:*



where E^a is the set of absent edges of a diagram in the left-hand side of the relation (it can be empty).

Lemma 5.7 *The type (d) faces are glued as soon as β is a constant on each isomorphism class of diagrams.*

Now, these lemmas imply Proposition 5.4 since the map β defined by (4.4) satisfies their conditions.

5.3 Gluing methods and orientations

Now we are going to define a general case of Ψ-isomorphisms between two faces $F(\Gamma, A)$ and $F(\Gamma', A')$ of a same type ((c) or (d)). The problem will be to determine their signs.

We suppose that $\Gamma/A = \Gamma'/A'$ (as labelled diagrams), and that the Ψ-isomorphism preserves the projection onto $C_{\Gamma/A}$. So, the only transformations

concern the graphs A and A' and the spaces $C(A)$ and $C(A')$ (Notation 2.2), and if $A \cap U$ and $A' \cap U'$ are nonempty, they are mapped to the same line l_s in the two identified configurations.

Except for the $\mathcal{R}$STU gluing, we shall suppose that $E_A = E_{A'}$. This implies that A and A' have the same respective numbers of univalent and trivalent vertices.

For convenience, we are going to label all vertices in A independently of the labelling of the edges, starting with the univalent vertices (this means the elements of U), if they exist. We denote them by x_i for $i \in X$, $X = \{0, \cdots, \#A - 1\}$. By convention, we put the vertex x_0 at the origin of the space R_A (thus, $x_0 \in U$ if $A \cap U \neq \emptyset$). So C^A, which contains $C(A)$, is identified with the space of half lines in the vector space $(\mathbb{R}^3)^{X-\{0\}}$. We do the same for Γ' (so the univalent vertices in A and A' are labelled by the same elements of X).

The neighbourhoods of the faces are locally identified with the fiber bundle $\mathcal{E}$ with base $C_{\Gamma/A}$ and fiber $\mathcal{E}_s$, where $\mathcal{E}_s$ is the space defined by: $\mathcal{E}_s = (\mathbb{R}^3)^{X-\{0\}}$ if $A \cap U = \emptyset$; and if $A \cap U \neq \emptyset$, $\mathcal{E}_s$ is the subspace of $(\mathbb{R}^3)^{X-\{0\}}$ consisting of the configurations which map $A \cap U$ to l_s, where s is defined by the base point in $C_{\Gamma/A}$.

We shall suppose that the bijection between the faces is defined over each element of $C_{\Gamma/A}$ by a linear transformation $\mathcal{L}$ of $\mathcal{E}_s$. If $A \cap U \neq \emptyset$, consider the set of vertices $A \cap U$ as a subset of X: we shall suppose that $\mathcal{L}$ is the identity on the corresponding $\mathbb{R}^3$ factors, and that the orientations of these vertices are preserved from Γ to Γ'.

Then, the bijection between the faces induces a bijection of their neighbourhoods which preserves any fixed orientation of $\mathcal{E}$ iff $\det \mathcal{L} > 0$ (which does not depend on s).

Now, let us deduce its effect on the orientations of C_Γ and $C_\Gamma{}'$ as defined in Subsection 4.2 : the local coordinates, first labelled by the vertices, are then labelled by the half-edges.

So, the labelled graphs A and A' define two bijections b and b' between

$$\{i \in X | x_i \in U\} \cup \{(i,1),(i,2),(i,3) | i \in X, x_i \in T\}$$

and the set $\cup A \subseteq E_2$ of half-edges which belong to a vertex in A. Then, the sign of the Ψ-isomorphism between $F(\Gamma, A)$ and $F(\Gamma', A')$ is the product of the sign of $\det \mathcal{L}$ with the signature of $b^{-1}b'$.

5.4 Proofs for the gluings

Proof of Lemma 5.5:

Each term of an $\mathcal{R}$IHX relation corresponds to a type (c1) face $F(\Gamma, e)$ where Γ is the labelled diagram of the term, and e is the edge involved in the relation.

If some type (c1) face does not fit into such a relation, it is because one other diagram of the IHX relation is not represented in $D_{n,k}(M)$, so it is not 3-connected to U and its faces are degenerated. Since all faces of this unwritten

$\mathcal{R}$IHX relation are Ψ-isomorphic, they are all degenerated. This is why no relation is necessary for them.

So we have a partition of at least all non-degenerated type (c1) faces, and what we have to check is that this partition is a gluing, where the gluing condition is the $\mathcal{R}$IHX relation.

In the above constructions, we can always take $\mathcal{L} = \mathrm{Id}$, preserve the graph structure of A but just use permutations of the set E'_A. This provides Ψ-isomorphisms of faces with sign the signature of this permutation (it must be considered modulo the transpositions of two edges attached to the same vertex of A, because these transpositions induce the identity).

The verification of the signs in the $\mathcal{R}$IHX relation is now easy. □

Proof of Lemma 5.7:

We consider now the non-degenerated type (d) faces.

According to Proposition 3.4, each edge in E'_A meets at least two edges (denoted as pairs of vertices) $e_1 = \{x, y\}$ and $e_2 = \{x, z\}$ in E_A. Suppose x has a non-zero label. Let $\mathcal{L}$ be the map which maps x to $y + z - x$, and preserves the other vertices. This transformation has determinant -1.

Then apply the only change of structure of A which fits in with this transformation: it is a mere relabelling, which exchanges e_1 and e_2 and reverses them: if e_1 goes from x to y in A, then it goes from z to x in A'. Suppose that these diagrams have the same orientation through this relabelling. This defines an even permutation of $\cup A$, so this Ψ-isomorphism is negative.

Now, here is a way to give a partition of these faces into pairs, which is a gluing as soon as β is constant on each isomorphism class of diagrams with the same orientation.

Consider an (equivalence) class of pairs (Γ, A) (concerned here) such that Γ/A is fixed as a labelled diagram, and the isomorphism class of the pair (Γ, A) is also fixed. Then fix an arbitrary edge $e \in E'_A$. By this choice, the diagrams in this class are grouped in pairs according to the above construction. □

Proof of Lemma 5.6:

First let us check that for each pair (Γ, A) where A is an edge or a pair of consecutive univalent vertices of Γ, the face $F(\Gamma, A)$ can be identified with the space $C'_{\Gamma/A}$: indeed, let us distinguish two cases.

If $E_A = \emptyset$, it comes from the fact that the configuration of A is determined by the one of Γ/A and the tangent map of L.

If $E_A \neq \emptyset$, we have $A \not\subseteq U$. This allows us to put the unique element of E_A among the absent edges.

Now, all faces of the form $F(\Gamma, A)$ which give a fixed Γ/A are Ψ-isomorphic, thus, they are all degenerated when one is. This is true for $k = 2$, thus also for the faces defined by the terms of an STU relation for any value of k. This proves that all non-degenerated type (c2) faces occur in an $\mathcal{R}$STU relation.

Let us study the signs of these Ψ-isomorphisms and check that the $\mathcal{R}$STU relations are the gluing conditions. The sign of the Ψ-isomorphism between both left-hand side diagrams comes from the same method as in the $\mathcal{R}$IHX gluing.

If the sum on the right-hand side is empty, this relation is verified for the expression (4.4) because it is deduced from the STU relation and the fact that the missing diagram was forced to cancel in $\mathcal{A}_n^k(M)$.

Otherwise, consider the following diagrams, where each half-edge is marked with the name of a component in a local coordinate system of the neighbourhood of each face. Here we use the spherical coordinates for the trivalent vertex, and u stands for the radial coordinate:

We see that these two labellings differ by the transposition (x, s). This proves that these two diagrams appear in the relation with different signs (or on different sides of the equality). □

6 Anomaly

6.1 The anomaly as a degree of a map

Recall that the non-degenerated type (a) and (a') faces are defined by a connected component A of Γ, whose configurations map all univalent vertices of A to a line l_s directed by the tangent vector s to L. To study them, we need to introduce a few definitions, where we replace M by the non-compact line $\mathbb{R}$. This does not change the fundamental notions of diagrams and configuration spaces ($D_{n,k}(\mathbb{R})$ is identical to $D_{n,k}(J)$ for instance).

Notation 6.1 Let $D_{n,k}^W$ be the set of connected elements of $D_{n,k}(\mathbb{R})$.

For all $s \in S^2$, let l_s be the standard map $u \mapsto ux$ from $\mathbb{R}$ to the oriented axis of $\mathbb{R}^3$ with direction s. By an abuse of notation, l_s will also mean this oriented axis.

For any $\gamma \in D_{n,k}^W$, let $W_s(\gamma)$ denote the quotient of $C_\gamma(l_s)$ by the translations and dilatations of $\mathbb{R}^3$ which preserve the axis l_s.

Then, let $W(\gamma)$ denote the union of the $W_s(\gamma)$ for $s \in S^2$.

We are going to give these spaces an orientation. First, orient the two-dimensional Lie algebra of the group of translations and dilatations which preserve l_s, by giving it a basis: take a first vector oriented to the positive translations of the oriented axis l_s, and a second vector oriented to the dilatations of ratio greater than one.

Now, since the space $C_\gamma(l_s)$ is oriented by the method of Subsection 4.2, this orients the quotient space $W_s(\gamma)$, and therefore $W(\gamma)$ using the standard orientation of the base space S^2. There is no ambiguity, since all considered spaces are even-dimensional.

Then we define a map Ψ_γ from $W(\gamma)$ to $(S^2)^{E^v}$, and a map Ψ'_γ from $W'(\gamma) = W(\gamma) \times (S^2)^{E^a}$ to $(S^2)^E$, exactly as before.

Now we are going to see that these spaces can be glued together according to the same formulas, except that there are no anomalous faces anymore.

Proposition 6.2 *For the map β defined by the same formula (4.4) on $D^W_{n,k}$, the linear combination of spaces*

$$W_{n,k} = \sum_{\gamma \in D^W_{n,k}} \beta(\gamma) W'(\gamma)$$

is completely glued, hence the union Ψ of the Ψ'_γ admits a degree on it.

To prove this proposition, we shall use the following lemma[4]

Lemma 6.3 *If L is a generic curve with support J, then for any $\gamma \in D^W_{n,k}$, we have* $\dim \Psi(W(\gamma)) \leq \dim \Psi(C_\gamma(L))$.

Proof:

Suppose that $\dim \Psi(W(\gamma)) > \dim \Psi(C_\gamma(L))$. Let F be the type (a) face of $C_\gamma(L)$: it is made of the fibers of $W(\gamma)$ over the curve c of S^2 which is the tangent map of L. This curve is generic since L is. We have

$$\begin{aligned} & \dim \Psi(F) \leq \dim \Psi(C_\gamma(L)) < \dim \Psi(W(\gamma)) \\ \Rightarrow\ & \dim \Psi(F) = \dim \Psi(C_\gamma(L)) = \dim \Psi(W(\gamma)) - 1 \\ \Rightarrow\ & \forall\, s \in S^2,\ \dim \Psi(F) = \dim \Psi(W_s(\gamma)) + 1 \end{aligned}$$

because if the image of a fiber $W_s(\gamma)$ did not sweep one dimension of the product of spheres when s runs over the generic curve c of S^2, then it could not sweep more dimension when s runs over the whole sphere.

Let $\mathcal{U}$ be a nonempty open subset of $\Psi(W(\gamma))$, and let π be some projection of $\mathcal{U}$ to an open subset $\mathcal{U}'$ of S^2, defined by: $\forall\, x \in \mathcal{U}$, $\pi(x)$ is the unique element s of $\mathcal{U}'$ such that $x \in \Psi(W_s(\gamma))$.

We can suppose that $\mathcal{U}'$ meets c because of the $SO(3)$-invariance. Then, since F is a face of $C_\gamma(L)$ and $\dim \Psi(F) = \dim \Psi(C_\gamma(L))$, there exists a nonempty open subset $\mathcal{U}''$ of $C_\gamma(L)$ near F such that $\Psi(\mathcal{U}'') \subseteq \mathcal{U}$ and $\pi(\Psi(\mathcal{U}'')) \subseteq c$.

Now we have a contradiction: for each configuration $f \in \mathcal{U}''$, $\pi(\Psi(f))$ informs us on a tangent vector of L. But such an information cannot be in f for the following reason: the only information on L that is contained in f is the datum of the finite set $f(U) \subseteq L(J)$, and one can slightly modify L so that $L(J)$ still contains $f(U)$ but $L'(J)$ avoids the point $\pi(\Psi(f))$. □

Proof of Proposition 6.2:

[4]This lemma is needed because of the way we defined the spaces $\mathcal{A}^k_n(M)$ and stated Proposition 1.11: if we change the conditions on codimensions into conditions of 3- and 4-connectedness for the diagrams, then this lemma can be dropped.

The method is to use the same proofs as those of Sections 4 and 5 in this new situation, where there are no equivalents of type (a), (a') and (c) faces (for the scale is relative and the diagrams are connected): we just have to check that the map Ψ defined on the spaces $W(\gamma)$ factorises in the same way on the faces. It is very easy to do for the faces $F(\gamma, A)$ such that $A \subseteq T$:

$$F(\gamma, A) = W(\gamma/A) \times C(A).$$

The other faces can also be seen as fibered spaces with bases $W(\gamma/A)$ and fiber $W_s(A)$: over each $y \in W(\gamma/A)$ we have the fiber $W_s(A)$ where $y \in W_s(\gamma/A)$. Lemma 6.3 is involved here to justify the $\mathcal{R}$STU gluing.

The only new thing to prove is that if an $\mathcal{R}$STU relation involves a non-connected diagram, then the faces are degenerated. This can easily be done by projecting the fiber over S^2 to the product of the corresponding fibers for the connected components: this product has one dimension less, corresponding to the relative dilatations between the configurations of the two subdiagrams. □

Definition 6.4 Let $\mathcal{D}^W$ be the set of connected elements of $\mathcal{D}(\mathbb{R})$. In other words, $\mathcal{D}^W$ is the set of isomorphism classes of diagrams in the $D^W_{n,2}$ when n runs over $\mathbb{N}$, together with a choice of arbitrary orientations. For all $\gamma \in \mathcal{D}^W$ we define the integral

$$f_\gamma = \int_{W(\gamma)} \Psi_\gamma^* \Omega$$

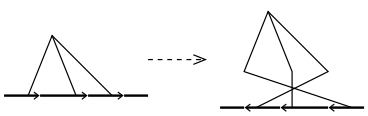
Then we define the anomaly to be

$$\alpha = \sum_{\gamma \in \mathcal{D}^W} \frac{f_\gamma}{2|\gamma|}[\gamma] \in \mathcal{A}(\mathbb{R}).$$

Note that the degree n part α_n is half the degree of the map Ψ of the above proposition, when $\mathcal{A}^k_n(M) = \mathcal{A}_n(M)$. Moreover, $\alpha_1 = \frac{[\theta]}{2}$. We are going to see that it is also the degree of a map defined on the quotient of $W_{n,k}$ by a symmetry, with still the same coefficients.

6.2 Symmetry properties of the anomaly

For any labelled oriented diagram γ with support $\mathbb{R}$, let $\bar{\gamma}$ be the diagram obtained by reversing the injection $i : U \longrightarrow \mathbb{R}$ (by the composition with the involution $u \mapsto -u$ of $\mathbb{R}$), and transporting the orientations of univalent vertices with U by this involution:



So the order of the vertices is changed. This defines a linear involution of each $\mathcal{A}_n(\mathbb{R})$ and an automorphism of $\mathcal{A}(\mathbb{R})$.

We say that a element of $\mathcal{A}(\mathbb{R})$ is S^1-*even* if it is preserved by this involution. It is unknown whether all elements of $\mathcal{A}(\mathbb{R})$ are S^1-even. Now, let $-\gamma$ denote the labelled diagram obtained from γ by reversing all its edges and preserving the vertex orientations. We obviously have $[-\gamma] = [\gamma]$ because these diagrams are isomorphic.

We have the two properties:

Lemma 6.5 *(i) For all $s \in S^2$ and $\gamma \in D^W_{n,k}$, there is a negative Ψ-isomorphism between $W_s(\gamma)$ and $W_{-s}(\bar{\gamma})$.*

(ii) the anomaly is S^1-even ($\bar{a} = a$).

Lemma 6.6 *(i) For all $s \in S^2$ and $\gamma \in D^W_{n,k}$, there is a Ψ-isomorphism between $W_s(\gamma)$ and $W_{-s}(-\gamma)$ with sign $(-1)^n$.*

(ii) The anomaly cancels at even degrees.

Proof of Lemma 6.5: The Ψ-isomorphism that we take from $W_s(\gamma)$ to $W_{-s}(\bar{\gamma})$ is the identity on configurations as maps from V to $\mathbb{R}^3$. The sign comes from the fact that we quotient the configurations by the action of the oriented group of dilatations and translations along the axis l_s which is reversed. Now, this map preserves the orientation between $W(\gamma)$ and $W(\bar{\gamma})$ because the antipodal map on the base space S^2 also reverses its orientation. Thus, $f_\gamma = f_{\bar{\gamma}}$. □

Proof of Lemma 6.6: The Ψ-isomorphism we take between $W_s(\gamma)$ and $W_{-s}(-\gamma)$ is the central symmetry on configurations. Note that this symmetry transports the orientation of l_s and the orientations of univalent vertices. Thus, the orientation changes come from the geometrical symmetry for each trivalent vertex, and the reversal of each edge in the diagram. So, the sign is $(-1)^n$ because of the formula $\#E - \#T = \deg\gamma = n$. Finally, the antipodal map on the base space and the isomorphism $-\gamma \sim \gamma$ give $f_\gamma = (-1)^{n+1} f_{-\gamma} = (-1)^{n+1} f_\gamma$: f_γ cancels for even n. □

Notation 6.7 When n is odd, the Ψ-isomorphism between $W_s(\gamma)$ and $W_{-s}(-\gamma)$ defines an orientation-preserving involution of $W_{n,k}$. We let $\widetilde{W}_{n,k}$ be the quotient space of $W_{n,k}$ by this involution, and let $\widetilde{D}^W_{n,k}$ be the quotient of $D^W_{n,k}$ by the involution $\gamma \mapsto -\gamma$. Thus,

$$\widetilde{W}_{n,k} = \sum_{\gamma \in \widetilde{D}^W_{n,k}} \beta(\gamma) W'(\gamma).$$

We can note that the image of α_n in $\mathcal{A}^k_n(\mathbb{R})$ is equal to the degree of the map Ψ defined on $\widetilde{W}_{n,k}$.

6.3 The labelled generalised diagrams and their configuration spaces

Now we are going to define the additional configuration spaces to complete C' in order to glue all the faces and define the degree of Ψ in the case of a link.

First, identify M with $X \times S^1$ where X is a finite set. Then define a family of smooth maps $\varphi_m : D^2 \longrightarrow S^2$ for $m \in X$, where each of them coincides with the tangent map to the link L on the boundary $\{m\} \times S^1$ of D^2. Here we equip the disc D^2 with the opposite orientation to the usual one. So, the basis (tangent oriented by S^1, outward normal) is direct.

Notation 6.8 For all $\gamma \in D^W_{n,k}$ and $m \in X$, let

$$\mathcal{W}_m(\gamma) = \varphi_m^* W(\gamma)$$

be the fibered space over D^2 which is the union of the $W_{\varphi_m(x)}(\gamma)$ for $x \in D^2$, and equip it with the product orientation.

In the same way as in Subsection 4.1, we let $E = \{e_1, \cdots, e_N\}$ be the set of labelled "edges" $e_i = \{2i-1, 2i\}$. We recall the formula $k = 3n - N$.

Now, we fix N and let n and k vary for the sets of diagrams $D^W_{n,k}$ and $\widetilde{D}^W_{n,k}$.

Precisely, let D^W_N denote the union of the $D^W_{n,3n-N}$ for all n. And let $\widetilde{D}^W_N$ denote the union of the $\widetilde{D}^W_{n,3n-N}$ for all n such that n is odd.

In fact, these unions concern the n such that $2n - 1 \leq N$, for connected diagrams of higher degrees would have more edges.

Definition 6.9 A *labelled generalised diagram* with support M is the data of a decomposition of E into a disjoint union

$$E^v \cup E^a \cup E^W$$

together with:

• a labelled diagram Γ_0 with support M such that its set of visible edges is E^v, in the sense of Definition 4.2;

• a set Y of diagrams in $D^W_N \sqcup \widetilde{D}^W_N$ such that their sets of visible edges form a partition of E^W, together with a map $\gamma \mapsto m(\gamma)$ from Y to X. We let $Y_1 = Y \cap D^W_N$ and $Y_2 = Y \cap \widetilde{D}^W_N$.

The *degree* of a labelled generalised diagram is the sum of the degrees of Γ_0 and all the diagrams $\gamma \in Y$. Let $D^g_{n,3n-N}$ denote the set of these generalised diagrams of degree n. An arbitrary orientation is chosen on each of them.

Note that $D_{n,k}(M) \subseteq D^g_{n,k}$. Indeed, $D_{n,k}(M)$ is the set of generalised diagrams $\Gamma \in D^g_{n,k}$ such that $Y = \emptyset$, or equivalently $E^W = \emptyset$.

Definition 6.10 Let the *configuration space* of a generalised diagram be

$$C_\Gamma = C_{\Gamma_0} \times \prod_{\gamma \in Y_1} \mathcal{W}_{m(\gamma)}(\gamma) \times \prod_{\gamma \in Y_2} W(\gamma).$$

with the product orientation. This is again completed with the help of the absent edges:

$$C'_\Gamma = C_\Gamma \times (S^2)^{E^a}.$$

The product orientation is well-defined because all spaces are even-dimensional. It is reversed when we change the orientation of a single vertex. The canonical map Ψ extends to the configuration spaces of generalised diagrams in the obvious way.

6.4 Gluing the anomalous faces

Proposition 6.11 *Let p be a fixed map from X to $\mathbb{Z}$. We extend β to generalised diagrams according to the same formula* (4.4) *where now $e_\Gamma = \#(E^v \cup E^W)$ and the images $[\Gamma] \in \mathcal{A}(M)$ of the generalised diagrams Γ are defined by*

$$[\Gamma] = [\Gamma_0] \prod_{\gamma \in Y_1} [\gamma]^{(m(\gamma))} \cdot \prod_{\gamma \in Y_2} p_{m(\gamma)} [\gamma]^{(m(\gamma))}. \tag{6.12}$$

Then, the canonical map Ψ defined on the linear combination

$$\sum_{\Gamma \in D^g_{n,k}} \beta(\Gamma) C'_\Gamma$$

admits a degree which belongs to the integral lattice generated by the elements $\beta(\Gamma)$ in the space $\mathcal{A}^k_n(M)$ where Γ runs over the diagrams in $\mathcal{D}_n(M)$ such that $\Psi(C_\Gamma(L))$ has codimension 0 for a generic L.

Proof:

Ordinary gluings.

The ordinary gluings of Γ_0 and all the $\gamma \in Y$ are an immediate consequence of Section 5 and Proposition 6.2. Here the $\mathcal{R}$STU relation always involves the set E^a of absent edges of the generalised diagram, given in Definition 6.9.

Anomaly gluings.

Now we are going to glue together the following two types of faces.

- The type (a) and (a') faces for the subdiagram Γ_0 of a generalised diagram;
- The faces induced by the boundary of D^2 on some $\mathcal{W}_{m(\gamma)}(\gamma)$.

These faces are glued in pairs in an obvious way, where one face of the first type is glued to a piece of a face of the second type delimited by the positions of the univalent vertices of Γ_0 on the corresponding component $\{m\} \times S^1$ of M.

Now we have to check that the two orientations are opposite in any case. The orientation of D^2 has been defined by the basis (tangent oriented by S^1, outward normal), whereas for the type (a) or (a') faces, we must consider the orientation of the Lie algebra by which we quotiented, when we defined the orientation of the spaces $W_s(\gamma)$: we took the basis (translations in the sense of s, growing dilatations). But s represents the generic tangent vector to L with the positive orientation, whereas the growing dilatations point inward to the configuration space.

Generators.

By definition, the degree of Ψ is equal to the count of preimages of a generic point with coefficients $\beta(\Gamma)$. So, it belongs to the integral lattice generated by the $\beta(\Gamma)$ when Γ runs over the generalised diagrams such that $\Psi(C_\Gamma)$ has nonempty interior. Then, we may keep the only non-generalised diagrams in this list, thanks to Lemma 6.3 and Formula 6.12. □

6.5 Proofs of Propositions 1.3 and 1.11

Notation 6.13 For each $m \in X$, define the integral

$$I_m = \int_{D^2} \varphi_m^* \omega.$$

Lemma 6.14 *The integrals on the spaces $\mathcal{W}_m(\gamma)$ obey the identity*

$$\int_{\mathcal{W}_m(\gamma)} \Psi_\gamma^* \Omega = f_\gamma I_m.$$

This comes from the invariance of the space $W(\Gamma)$ and of the volume form Ω by rotations of S^2. □

Proposition 6.15 *The expression* $\deg \Psi$ *of Proposition 6.11 is the projection in $\mathcal{A}_n^k(M)$ of*

$$Z \prod_{m \in X} \exp\left((p_m + 2I_m)\alpha^{(m)}\right).$$

The proof of this proposition is just a matter of calculus. Let $\mathcal{D}_{n,k}^g$ be the set of isomorphism classes of generalised diagrams in $D_{n,k}^g$. It is the set of classes of degree n generalised diagrams Γ such that $e_\Gamma \leq N$.

We have

$$\begin{aligned}\deg \Psi &= \sum_{\Gamma \in D_{n,k}^g} \frac{(N - e_\Gamma)!}{N!\ 2^{e_\Gamma}} I_L(\Gamma)[\Gamma] \\ &= \sum_{\Gamma \in \mathcal{D}_{n,k}^g} \frac{I_L(\Gamma_0)[\Gamma_0]}{|\Gamma|} \prod_{\gamma \in Y_1} f_\gamma I_{m(\gamma)}[\gamma]^{(m(\gamma))} \cdot \prod_{\gamma \in Y_2} p_{m(\gamma)} f_\gamma [\gamma]^{(m(\gamma))}\end{aligned}$$

For any generalised diagram Γ and any $\gamma \in \mathcal{D}^W$, let $\varepsilon_\gamma^\Gamma$ denote the number of copies of γ in the set Y_1 of Γ, and let η_γ^Γ denote the number of copies of γ in its set Y_2.

Then, the number of automorphisms of Γ is

$$|\Gamma| = |\Gamma_0| \prod_{\gamma \in \mathcal{D}^W} |\gamma|^{\varepsilon_\gamma^\Gamma} (\varepsilon_\gamma^\Gamma)!\ (2|\gamma|)^{\eta_\gamma^\Gamma} (\eta_\gamma^\Gamma)!$$

So we have

$$\deg \Psi = \sum_{\Gamma \in \mathcal{D}_{n,k}^g} \frac{I_L(\Gamma_0)[\Gamma_0]}{|\Gamma_0|} \prod_{\gamma \in \mathcal{D}^W} \frac{(I_{m(\gamma)})^{\varepsilon_\gamma^\Gamma}}{\varepsilon_\gamma^\Gamma!} \left(\frac{f_\gamma [\gamma]^{(m(\gamma))}}{|\gamma|}\right)^{\varepsilon_\gamma^\Gamma + \eta_\gamma^\Gamma} \frac{1}{\eta_\gamma^\Gamma!} \left(\frac{p_{m(\gamma)}}{2}\right)^{\eta_\gamma^\Gamma}$$

This is the image in $\mathcal{A}_n^k(M)$ of the degree n part of

$$\left(\sum_{\Gamma_0 \in \mathcal{D}} \frac{I_L(\Gamma_0)[\Gamma_0]}{|\Gamma_0|}\right) \prod_{m \in X} \exp\left(\sum_{\gamma \in \mathcal{D}^W} \frac{f_\gamma [\gamma]^{(m)}}{|\gamma|} \left(I_m + \frac{p_m}{2}\right)\right)$$

which is nothing but

$$Z \prod_{m \in X} \exp\left((p_m + 2I_m)\alpha^{(m)}\right).$$

□

Lemma 6.16 *For all m, $I_L(\theta_m) + 2I_m \in \mathbb{Z}$.*

We can restrict ourselves to one fixed component m of M, thus to the case of a knot. Apply Proposition 6.15 to the case $n = N = 1$: the degree 1 part of $Z\exp((p_m + 2I_m)\alpha)$ is $I_L(\theta)\frac{[\theta]}{2} + (p_m + 2I_m)\frac{[\theta]}{2}$. It is the degree of a map where the coefficient is $\beta(\theta) = \frac{[\theta]}{2}$. Thus, $I_L(\theta_m) + 2I_m$ is an integer. □

Proof of Proposition 1.3:

The above constructions work for any map p from X to $\mathbb{Z}$. Now, let $p_m = -(I_L(\theta_m) + 2I_m)$. In an isotopy, we let the maps φ_m vary continuously, so the integer p_m is constant for any m.

Now let

$$Z^0 = Z \prod_{m \in X} \exp\left((p_m + 2I_m)\alpha^{(m)}\right).$$

Its part of each degree is the degree of a map, thus it is constant in each isotopy class. We have $p_m + 2I_m = -I_L(\theta_m)$. We conclude that

$$Z = Z^0 \prod_{m \in X} \exp\left(I_L(\theta_m)\alpha^{(m)}\right).$$

□

Proof of Proposition 1.11:

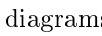
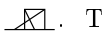
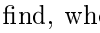
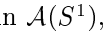
We know that $I_L(\theta_m) + 2I_m$ is an integer. If we suppose that $I_L(\theta_m)$ is an integer, then we can let $p_m = -2I_m$. Thus, for any fixed n and k, Z_n^k is equal to $\deg \Psi$. The conclusion comes from Proposition 6.11. □

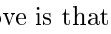
Remark 6.17 It is well-known that for a knot ($I_m = I$, $\theta_m = \theta$, $\varphi_m = \varphi$), the Gauss integral $I_L(\theta)$ is some sort of measure of the framing of the knot, that is, the linking number of the knot with a parallel curve. Now we have proved that $I_L(\theta) + 2I$ is equal to an integer. So one can wonder how to construct geometrically the parallel curve with framing $I_L(\theta) + 2I$ from the map φ. Here is the answer:

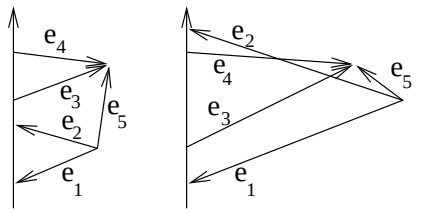
Consider the fibre bundle over D^2, with fiber over $x \in D^2$ the normal plane to the vector $\varphi(x)$. Since D^2 is homotopically trivial, there is a unique trivialisation of this bundle up to homotopy. This gives a homotopy class of trivialisation of the bundle over the boundary S^1 of D^2, which is the normal bundle to the knot. Thus, it gives our parallel curve up to isotopy.

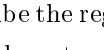
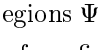
7 Vanishing of the anomaly in degree 3

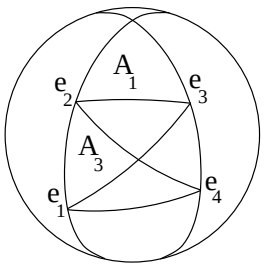
In this section we are going to prove that the degree 3 part of α vanishes, although the integrals f_γ that occur in its expression do not individually cancel.

It is easy to see that the degree 3 diagrams $\gamma \in \mathcal{D}^W$ are ,  , and  . They have no automorphisms other than the identity. We also easily find, when considering their classes in $\mathcal{A}(S^1)$, that [ ] $= -$[ ] and that [] $= 0$. So, what we have to prove is that f $= f$  .

Put the following labelling on the diagrams and  :



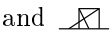
Now we are going to describe the regions $\Psi(C$  $)$ and $\Psi(C$  $)$ of $(S^2)^E$, $E = \{e_1, \cdots, e_5\}$. They are the sets of configurations of E in S^2 which verify the two conditions:



1) The points e_1, e_2, e_3, e_4 form a "square", that is, they obey the four conditions obtained by circular permutations of this one: the points e_3 and e_4 are on the same side of the vector plane $(e_1\ e_2)$which contains e_1 and e_2. (This implies that e_1 and e_3 are on different sides of the diagonal $(e_2\ e_4)$.)

2) The point e_5 is in the region A_1 in the case of , and in the region A_3 in the case of .

Note that there are two possible orientations for the square (e_1, e_2, e_3, e_4), corresponding to the two sides of the vector plane $(e_1\ e_2)$ where the points e_3 and e_4 can be. This splits the configuration spaces into two subsets. We can restrict the study to one of them, since the other one is obtained by the central symmetry: we know that this symmetry preserves the integral according to Lemma 6.6. On the interior of each of these subsets, Ψ is a diffeomorphism.

In fact, these canonical maps Ψ for the two diagrams and  are two restrictions of the diffeomorphism Ψ defined:

From the space of all "configurations" of this diagram on an axis which are not contained in a plane, injective on each edge, where the order between the

univalent vertices of e_1 and e_2 is respected, and the same for e_3 and e_4; but with no other order conditions between these vertices (it is still quotiented by dilatations and translations);

To the region of $(S^2)^E$ defined by the conditions that e_1, e_2, e_3, e_4 form a square, and that e_5 is inside the biangle with one side containing e_1 and e_2, and the other side containing e_3 and e_4.

Now, the jacobian of Ψ does not cancel between the areas A_1 and A_3, so its sign remains unchanged: f and f have the same sign.

Finally, they have the same absolute value because the two regions $\Psi(C$ $)$ and $\Psi(C$ $)$ are described by analogous systems of inequalities:

$\Psi(C$ $)$ is the set of $(e_1, e_2, e_3, e_4, e_5)$ such that (e_1, e_2, e_3, e_4) forms a square and e_5 is inside this square, in the triangle limited by the diagonals and the side (e_1, e_2).

$\Psi(C$ $)$ is the set of $(e_1, e_2, e_3, e_4, e_5))$ such that $(e_2, e_3, -e_1, -e_4)$ forms a square and e_5 is inside this square, in the triangle limited by the diagonals and the side (e_2, e_3). □

8 Z^0 is a universal Vassiliev invariant

We are going to see that the invariant Z^0 is a universal Vassiliev invariant.

We recall the defining property of universal Vassiliev invariants $\mathcal{I}$: for any integer n and any singular oriented link with n double points, that is, an immersion L of M into $\mathbb{R}^3$ which is an imbedding except for a set X of n points which have two preimages, the alternate sum of desingularised links

$$\sum_{\zeta : X \longrightarrow \{-1,1\}} \left(\prod_i \zeta(i) \right) \mathcal{I}(L_\zeta)$$

is zero in all degrees $< n$ and is equal to the chord diagram defined by the positions of singular points in M in degree n. Here, L_ζ is the link obtained from L by replacing the singularities by the crossings with signs defined by ζ.

To prove this proposition, we are going to see that Z itself has this property, provided that the movement of desingularisation is infinitesimal.

Consider Z in terms of configured diagrams as described in Subsection 4.4. The spaces C of the different desingularised links are in correspondence except at places where some part of graph connects the two strands in the neighbourhood of a desingularisation. Then, the alternate sum cancels every contributions of the parts of the C in which there is not the presence of such parts of graph near *every* singular point.

The configured diagrams which do contain such parts of graph near every singular points must have a degree $\geq n$: indeed, they must have at least one univalent vertex on each strand of the singular points, so globally at least $2n$ univalent vertices. The only degree n diagram which has this property is precisely the expected chord diagram.

Now, the alternate sum of integrals on these configured chord diagrams is easily computed: at every neighbourhood of a singular point, the direction of the chord runs through a hemisphere limited by the tangent plane at the singularity for each desingularised diagram. The difference gives the surface of the whole sphere which is equal to 1 according to our definitions. □

9 Extension to tangles

9.1 The first convergence result

The definition of the limits $C^l_\Gamma(L)$ of the configuration spaces $C_\Gamma(h_\lambda \circ L)$ when $\lambda \to 0$ will be particular cases of the following general construction:

Definition 9.1 Let $(\mathcal{F}_\lambda)_{\lambda\in]0,1]}$ be a semi-algebraic, one-parameter family of subsets of a fixed compact, semi-algebraic manifold H together with a smooth, semi-algebraic map Ψ from H to $(S^2)^E$, such that the maximal dimension of $\mathcal{F}_\lambda$ is $\leq (2\#E)$. Then we define the limit $\overline{\mathcal{F}}$ of $\mathcal{F}_\lambda$ as the intersection of $\{0\} \times H$ with the closure of the set of $(\lambda, x) \in]0,1] \times H$ such that $x \in \mathcal{F}_\lambda$. This limit space must be counted with the limit orientation and algebraic multiplicities, as a part of an algebraic boundary.

In practice, in all the applications of this proposition in this article, there is no multiplicity problem: the limit space $\overline{\mathcal{F}}$ is everywhere the limit of only one contribution of $\mathcal{F}_\lambda$.

Proposition 9.2 *The integral of $\Psi^*\Omega$ on $\overline{\mathcal{F}}$ is equal to the limit of the integrals of $\Psi^*\Omega$ on the $\mathcal{F}_\lambda$s when λ approaches zero.*

Thus, the limit value $Z^l(L)$ of $Z(h_\lambda \circ L)$ is equal to the expression obtained by replacing the configuration spaces by their limits $C^l_\Gamma(L)$.
Proof of Proposition 9.2:

Recall that the integral of $\Psi^*(\Omega)$ on a space $\mathcal{F}_\lambda$ can be seen as the integral over the image space $(S^2)^E$ of the count (with signs) of preimages.

For the map Ψ defined on the set of $(\lambda, x) \in]0,1] \times H$ such that $x \in \overline{\mathcal{F}_\lambda}$, there is, according to [BCR], a partition of $(S^2)^E$ into a finite number of semi-algebraic subsets F, either open or with empty interior, such that $\Psi^{-1}(F)$ is Ψ-homeomorphic to a direct product $F \times G$. Let F be open. G is at most 1-dimensional; its isolated points do not contribute to our integral when λ approaches 0. It is easy to see that the limit integral over F of the count of preimages in the $\overline{\mathcal{F}_\lambda}$ is the volume of F, times the number (with signs) of the intervals in G which are not closed; it is equal to the integral over F of the constant count of preimages in the limit space. □

9.2 A general argument with dimensions

In the proofs of the propositions, we shall use several times the following argument to minorate the codimension of the images by Ψ of certain subsets of $\mathcal{H}$. Recall Notation 2.2 and 2.3 for $\mathcal{H}$ and its strata $C(G, S)$.

Let Γ be a connected diagram in $\mathcal{D}$. Let $f \in \mathcal{H}$ be the image in $\mathcal{H}$ of some limit configuration of this diagram. Then f belongs to some stratum $C(G,S)$ of $\mathcal{H}$. We are going to consider the configurations f_V of the diagram V/S. Let $V/S = U' \sqcup V'$ where U' is the image of U and T' is made of vertices coming only from T. For any $x \in V/S$, let n_x be the number of edges which contain x. Since Γ is connected and 3-connected to U, we have $n_x - 1 \geq 0$ for any $x \in U'$ and $n_x - 3 \geq 0$ for any $x \in T'$. Let E' be the set of edges of V/S.

The elements of U' will naturally be forced to lay on some curve l of $\mathbb{R}^3$. Let n_l be the dimension of the group $\mathcal{G}$ of dilatations-translations of $\mathbb{R}^3$ which preserve l. Let C be the set of maps from $T' \cup U'$ to $\mathbb{R}^3$ which map U' to l. Then the codimension of the image of the canonical map Ψ from C to $(S^2)^{E'}$ is minorated by

$$\begin{aligned}\dim(S^2)^{E'} - \dim(C/\mathcal{G}) &= 2\#E' + n_l - 3\#T' - \#U' \\ &= n_l + \sum_{x \in T'} (n_x - 3) + \sum_{x \in U'} (n_x - 1).\end{aligned}$$

This implies for example that if the codimension is zero, then the elements of T' are trivalent, those of U' are univalent and $n_l = 0$.

9.3 The separation results

Recall that $\overline{C_\Gamma} \subseteq \mathcal{H}' = \mathcal{H} \times M^U$ (Subsection 3). Then, $C^l_\Gamma(L)$ is also defined as a subset of $\mathcal{H}'$.

Proposition 9.3 *Let L_z denote the vertical component of L, and fix a diagram $\Gamma \in \mathcal{D}$. Except for a subset of $C^l_\Gamma(L)$ whose image by Ψ has no volume, the elements $x = (f, f') \in C^l_\Gamma(L)$ verify the following conditions: for the set U_1 of univalent vertices of any connected component of Γ we have*

1) $L_z \circ f'(U_1)$ is a singleton.

2) If $L_z \circ f'(U_1)$ is a critical value of L_z then $f'(U_1)$ is reduced to a critical point of L_z.

3) If $L_z \circ f'(U_1)$ is not a critical value of L_z then $\#f'(U_1) \geq 2$.

4) If the univalent vertices of two connected components of Γ are mapped by $L_z \circ f'$ to the same point, then this point is a critical value of L_z.

Except for the proof of 4), we may assume that Γ is connected. Thus, $U_1 = U$. We may also assume that $\bar{U} = V$, for the same reason as in Proposition 3.7.

Proof of 1) and 3):

Let $(f, f') \in C^l_\Gamma(L)$. We suppose that $L_z \circ f'(U)$ has at least two elements, or $f'(U)$ is a singleton which is not a critical point of L_z. In both cases, the configuration f_V maps all univalent vertices to the same vertical line. Then, we deduce from Subsection 9.2 that $\Psi(f)$ lies in a space with codimension at least two which is independent of f. □

We deduce from 1) that when $\#f'(U) \geq 2$, the horizontal components of f restricted to U coincide with those of $L \circ f'$ up to translations and dilatations. So, f_V maps all elements of U to the finite union of vertical lines which have the same intersection with the horizontal plane with altitude $L_z \circ f'(U)$, as $L(M)$.

Proof of 2): If $L_z \circ f'(U)$ is a critical value of L_z and $f'(U)$ is not reduced to a critical point of L_z, then the configuration f_V must be considered modulo the vertical translations. But there are finitely many critical values of L_z, so $\Psi(f)$ lies in a codimension 1 space.

Proof of 4): If it was not a critical value, then the configuration must be considered modulo independent vertical translations of the two components, which are compensated by the only one parameter of variation $L_z \circ f'(U_1)$. □

Proof of Proposition 1.15:

Consider $Z^l(L)$ in terms of configured diagrams (Subsection 4.4): it is an integral over the space of all limit configured diagrams Γ on L. According to Proposition 9.3 1), this space can be identified with the space of pairs (Γ', Γ'') of configured diagrams on L' and L'' respectively.

The fact that $Z^l(L) = Z^l(L')Z^l(L'')$ comes from the fact that the expressions $Z^l(L)$ and $Z^l(L')Z^l(L'')$ are integrals of identifiable spaces with the same density (for $[\Gamma] = [\Gamma'] \cdot [\Gamma'']$) □

9.4 The isotopy invariance

Proof of Proposition 1.17 (i):

Recall that for $n \geq 1$, the expression Z_n is equal to the integral of $\Psi^*\Omega$ over the glued formal linear combination of spaces

$$C(L) = \sum_{\Gamma \in D_{n,k}(M)} \beta(\Gamma) C'_\Gamma(L).$$

The boundary of this space is made of the anomalous faces, the extra faces and some degenerated faces.

Now, for each $\lambda \in]0,1]$, define the $(2N+1)$-dimensional fibered space $\mathcal{F}_\lambda$ with base $[0,1]$, and where the fiber over each $t \in [0,1]$ is the space $C(h_\lambda(\phi(t,\cdot)))$. The canonical map Ψ is well-defined and smooth on $\mathcal{F}_\lambda$, so we can apply the Stokes theorem to the closed form $\Psi^*\Omega$: the sum of integrals of $\Psi^*\Omega$ on the faces of $\mathcal{F}_\lambda$ cancels. These faces are:

- the two faces defined by $t = 0$ and $t = 1$: the integrals on them are $-Z_n(L)$ and $Z_n(L')$;
- the faces made of the faces of the $C'(h_\lambda(\phi(t,\cdot)))$ when t runs over $]0,1[$; but the ones from the degenerated faces do not contribute to the integral, for their images by Ψ are a one-parameter union of codimension 2 spaces, thus have codimension at least one.

This relation passes to the limit according to Proposition 9.2, and will give the expected relation $Z^l_n(L) = Z^l_n(L')$ provided that the limits of the other faces are mapped to fixed codimension 1 subsets of $(S^2)^E$.

In an anomalous face, a set of edges takes a position in $\Psi(W_s(\gamma))$ for some γ and some tangent vector s to L. But $\Psi(W_s(\gamma))$ is independent of L and has codimension 2, so let us check that the two-dimensional set of tangent vectors to all the tangles $h_\lambda(\phi(t,\cdot))$ for variable t converges to a dimension 1 limit in S^2 when λ approaches zero. Indeed, since M is compact, the non-vertical limit tangent vectors have the longitudes given by the values of the horizontal tangent vectors of $\phi(t,\cdot)$. The set of values of the horizontal tangent vectors is finite by assumption, so the limit set of the tangent vectors is contained in the union of the poles and a finite set of meridians. This proves the cancellation of the integral on the limits of the anomalous faces.

Now take an extra face: let V_1 be the connected component of the vertex which reaches the boundary of M, and let (f, f') belong to the limit of this face.

According to the proof of Proposition 9.3 1), we can suppose that $L_z \circ f'(U \cap V_1) = \{z\}$, where z is one of the boundary altitudes a or b. Then, since no tangent vector to $L(\partial M)$ is horizontal, f_{V_1} belongs to the closure of $C_{V_1/S}(x \times Id_{\mathbb{R}})$ where x is the planar configuration of $\phi(t, M) \cap (\mathbb{R}^2 \times \{z\})$. This space $C_{V_1/S}(x \times Id_{\mathbb{R}})$ has a codimension 1 image, so we just have to check that the set of these x is finite during the isotopy. x is the constant configuration $\phi(t, \partial M) \cap (\mathbb{R}^2 \times \{z\})$, plus possible extrema reaching the boundary altitude z, according to our definition 1.13 of a tangle. Up to changing the isotopy, we can suppose that no extremum reaches a boundary altitude for $t \in]0,1[$. □

The second part of Proposition 1.17 will be proved in the next section.

9.5 Connection and holonomy

Notation 9.4 Let X be a finite set with at least two elements. We denote by $\mathcal{C}_X$ the space of injections of X into the plane $\mathbb{R}^2$ modulo dilatations and translations.

Let $\mathcal{D}^c(X)$ be the set of the nonempty connected diagrams in $\mathcal{D}(X \times \mathbb{R})$ for which all univalent vertices do not go to the same component of $X \times \mathbb{R}$.

Let $\mathcal{P}(X)$ be the closed subspace of $\mathcal{A}(X \times \mathbb{R})$ which is the product of the spaces generated by the elements of $\mathcal{D}^c(X)$ at each degree. This space $\mathcal{P}(X)$ is stable under the Lie bracket $[x, y] = x \cdot y - y \cdot x$, thus it is a Lie algebra.

For a configuration $x \in \mathcal{C}_X$, we define the embedding $x \times \mathrm{Id}_{\mathbb{R}}$ of $X \times \mathbb{R}$ into $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. For any $\Gamma \in \mathcal{D}^c(X)$, let $C_\Gamma(x)$ denote the configuration space $C_\Gamma(x \times \mathrm{Id}_{\mathbb{R}})$ quotiented by the vertical translations (it has dimension $2e_\Gamma - 1$).

Let the space C^{tot}_Γ denote the fibered space over $\mathcal{C}_X$ such that the fiber over $x \in \mathcal{C}_X$ is $C_\Gamma(x)$. We still define the canonical map Ψ from C^{tot}_Γ to $(S^2)^E$.

Let $\omega(X)$ be the 1-form on $\mathcal{C}_X$ with values in $\mathcal{P}(X)$ defined by

$$\omega(X) = \sum_{\Gamma \in \mathcal{D}^c(X)} \frac{\omega_\Gamma[\Gamma]}{|\Gamma|}$$

where ω_Γ is the integral of $\Psi^*\Omega$ along the fibers of C^{tot}_Γ.

We must specify the orientations to define this form completely: we define the orientation of $C_\Gamma(x \times \mathrm{Id}_\mathbb{R})$ in the same way as in Subsection 4.2, and we substitute the dimension of a tangent vector to $\mathcal{C}_X$, to the dimension of the vertical translations by which we quotiented.

Recall that a braid is a tangle with no horizontal tangent vector, and it can also be seen as a path in $\mathcal{C}_X$. Since $\omega(X)$ is a 1-form on $\mathcal{C}_X$ with values in the Lie algebra $\mathcal{P}(X)$, we shall call it a "connection" (as in the construction of the Kontsevich integral). This allows us to construct the "holonomy" of this connection along a braid. We shall prove the following proposition:

Proposition 9.5 *(i) The integral $Z^l(\mathcal{B})$ for a braid $\mathcal{B}$ is equal to the holonomy of the connection $\omega(X)$ along the path $\mathcal{B}$.*

(ii) $\omega(X)$ is a flat.

(i) means that, if $\mathcal{B}$ is a smooth map from $[0,1]$ to $\mathcal{C}_X$ and $\mathcal{B}_t$ is its its restriction to $[0,t]$, the map $t \mapsto Z^l(\mathcal{B}_t)$ is characterised by the formulas

$$Z^l(\mathcal{B}_0) = 1$$
$$\frac{d}{dt} Z^l(\mathcal{B}_t) = Z^l(\mathcal{B}_t) \cdot \langle \omega(\mathcal{B}(t)), \frac{d}{dt}\mathcal{B}(t) \rangle$$

It is well-known [Ch] that the holonomy of a connection can also be formulated as a sum of multiple integrals on simplexes of the form $0 < t_1 < \cdots < t_i < 1$. But this is precisely the form of the expression $Z^l(\mathcal{B})$ according to Proposition 9.3 1) and 4): the values of t_i are the altitudes $L_z \circ f'(U_i)$ for the different connected components of Γ.

(ii) immediately comes from the isotopy invariance of Z^l.

10 Extension to q-braids

10.1 Compactification of the planar configuration space

Now we are going to compactify the space $\mathcal{C}_X$. If $B \subseteq X$, let $\mathcal{C}'_B$ be the space of nonconstant maps from B to the plane $\mathbb{R}^2$ modulo the dilatations and translations. It is a compact manifold. We have a canonical imbedding:

$$\mathcal{C}_X \hookrightarrow \prod_{B \subseteq X, \# B \geq 2} \mathcal{C}'_B.$$

This provides a compactification $\overline{\mathcal{C}}_X$ of $\mathcal{C}_X$.

A stratum of this compactification is labelled by a set $\mathcal{S}$ of subsets of X such that $X \in \mathcal{S}$, any B in $\mathcal{S}$ has at least two elements and any two elements of $\mathcal{S}$ are either disjoint or one included in the other. It is parametrised by the space

$$\mathcal{C}_{X,\mathcal{S}} = \prod_{B \in \mathcal{S}} \mathcal{C}_{B/\mathcal{S}}$$

where $B/\mathcal{S}$ is the quotient set of B by the greatest elements of $\mathcal{S}$ strictly included in B.

We define the fibered space $\overline{C}_\Gamma^{tot}$ over $\overline{\mathcal{C}}_X$ to be the closure of C_Γ^{tot} in the compact space $\overline{\mathcal{C}}_X \times \mathcal{H}'$. This fibered space $\overline{C}_\Gamma^{tot}$ need not be locally trivial above the corners of $\overline{\mathcal{C}}_X$.

Now we are going to study the restriction $\omega(X, \mathcal{S})$ of the connection $\omega(X)$ to $\mathcal{C}_{X,\mathcal{S}}$. This restriction is defined by integration along the fibers of the fibered space which is the restriction of $\overline{C}_\Gamma^{tot}$ over $\mathcal{C}_{X,\mathcal{S}}$.

10.2 Decomposition of the connection at the boundary

The duplication property of Z^l (Proposition 1.18) can easily be deduced from the following lemma, when one considers these integrals in terms of configured diagrams (Subsection 4.4).

Lemma 10.1 *For any path $\mathcal{B}$ in $\overline{\mathcal{C}}_X$ and any diagram $\Gamma \in \mathcal{D}^c(X)$, the integral of $\Psi^*\Omega$ over the configuration space $\mathcal{B}^*\overline{C}_\Gamma^{tot}$ equals its integral over the subset of $\mathcal{B}^*\overline{C}_\Gamma^{tot}$ made of the $(x, f, f') \in \overline{\mathcal{C}}_X \times \mathcal{H} \times M^U$ such that f belongs to the main stratum $C(G) = C(G, \{V\})$.*

Proof:

What we have to check is that the integral of $\Psi^*\Omega$ over the other strata ($S \neq \{V\}$) cancels.

The argument of Subsection 9.2 clearly implies that $\Psi^*\Omega$ vanishes there unless each image point of a univalent vertex in V/S is connected to the rest of the diagram by only one edge. In this case, if the subdiagram mapped to such a point is non-trivial, it must contain one bivalent vertex. Then, the central symmetry of this bivalent vertex relatively to the middle of the two vertices connected to it (as in the proof of Lemma 5.7), preserves the structure of the graph and reverses the orientation. So, the integrals on such configurations globally cancel. □

10.3 The associator

Lemma 10.2 *For all Γ, the integral of ω_Γ along a semi-algebraic path in $\overline{\mathcal{C}}_X$ converges.*

Proof: The integral of ω_Γ along a semi-algebraic path in $\overline{\mathcal{C}}_X$ is the integral of $\Psi^*\Omega$ on the submanifold of $\overline{C}_\Gamma^{tot}$ made of the fibers over the path. This is a semi-algebraic manifold, so this integral is equal to the integral in the product of spheres, of the count of preimages by Ψ. This count of preimages is a semi-algebraic map, thus it is bounded and the integral converges. □

Therefore, the holonomy also converges.

Proof of Proposition 1.20:

The pentagon and hexagon axioms are a direct consequence of the multiplicativity, the isotopy invariance and the duplication property of Z^l. The formula $\Phi^{-1} = \Phi^{(321)}$ comes from the invariance of Z during the symmetry around a vertical axis. The formula $\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$ results from the fact that the braid which defines Φ becomes the identity path in $\mathcal{C}_{\{1,2\}}$, $\mathcal{C}_{\{1,3\}}$ and $\mathcal{C}_{\{2,3\}}$ when we delete the other strand. □

10.4 The braiding and the anomaly

Proof of Lemma 1.21. According to Proposition 1.3, we have

$$Z^l\left(\right) = Z(O)\exp(-\mu\alpha)$$

for some $\mu \in \mathbb{R}$, where O is the circle. Then, since the algebra structure of $\mathcal{A}(S^1)$ is commutative,

$$Z^l\left(\right) = Z^l(\cap)^2(R \cdot 1_\cap) = Z(O)(R \cdot 1_\cap).$$

Thus,

$$R \cdot 1_\cap = \exp(-\mu\alpha).$$

Finally, we obtain $\mu = 1$ by considering the degree 1 part of this formula:

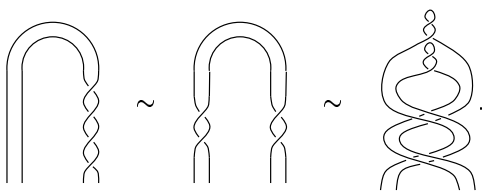
$$\frac{H}{2} \cdot 1_\cap = -\mu\frac{[\theta]}{2}.$$

The minus sign comes from the fact that the orientation of one univalent vertex changes from to . □

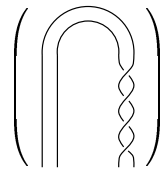
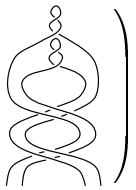
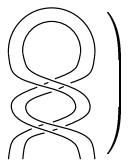
Proof of Proposition 1.17 (ii):

Moving a horizontal tangent vector by an angle of $\mu\pi$ radians is equivalent to inserting things like in a small neighbourhood near its position, where the angle between the directions of intersection of with horizontal planes from the bottom to the top is $\mu\pi$ radians. Then, for the same reason as Lemma 1.21, $Z^l( ) = \exp(-\mu\alpha)Z^l(\cap)$. We conclude thanks to the duplication property of Z^l. □

Proof of Proposition 1.22: Consider the following isotopy:



According to Proposition 1.17 and Lemma 1.21, we have

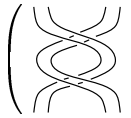
Z^l  $= Z^l$  $= \exp(-2\alpha^{(1)} - 2\alpha^{(2)}) Z^l$  .

Now, let ρ be the element of $\mathcal{A}(\{1,2\} \times J)$ such that

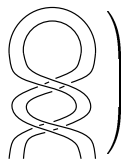
Z^l $= \rho \cdot 1$.

Here ρ is simply included in $\mathcal{A}(\{1,2\} \times J) \subseteq \mathcal{A}(\{1,2,3,4\} \times J)$.

According to the duplication property of Z^l, we have

Z^l  $= \Delta_1\Delta_2(R^{-2})$.

Since elements of a space $\mathcal{A}(M_j^Y)$ commute with the elements of $\mathcal{A}(Y \times J)$, we have

Z^l  $= \rho \cdot \Delta_1\Delta_2(R^{-2}) \cdot 1$ $= \rho \cdot \Delta(R^{-2} \cdot 1_\cap)$.

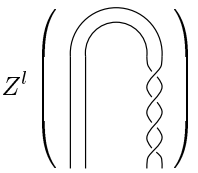
But

$$R^{-2} \cdot 1_\cap = \exp(2\alpha)$$

thanks to Lemma 1.21 and because the product by $1_\cap$ is a morphism of algebras from $\mathcal{A}(\{1,2\} \times J)$ to $\mathcal{A}(\{1\} \times J)$. Finally,

$$\Delta(\exp(2\alpha)) = \exp(2\Delta(\alpha))$$

because the map Δ from $\mathcal{A}(\{1\} \times J)$ to $\mathcal{A}(\{1,2\} \times J)$ is also a morphism of algebras. Now we can conclude: we have

Z^l  $= \rho \cdot (R^4)^{(34)} \cdot 1$.

Then, we just simplify by ρ and find

$$R^4 = \exp(-2\alpha^{(1)} - 2\alpha^{(2)}) \exp(\Delta(2\alpha)).$$

The degree zero part of R is 1, so we can apply the development of the fourth root to obtain the result:

$$R = \exp \frac{\Delta(\alpha) - \alpha^{(1)} - \alpha^{(2)}}{2}.$$

□

A Changing the map β

In this section, we study the possibility of choosing a map β different from Formula (4.4). It is of no use in the rest of this article.

A.1 Weakening the gluing conditions

Finding other maps is something difficult. We shall not see general solutions but we shall mention some elements of the problem.

The first aspect of the problem is to make a complete list of configuration spaces. We have seen that the configuration spaces of diagrams as we first defined has a codimension 0 image only if it is 4-connected to U. In fact this condition is also sufficient in the case of a generic knot or link (this is not true for the spaces $W(\gamma)$) but it is not very easy to prove and we shall not prove it here.

On the other hand, the configuration space of a diagram may have a codimension 2 image even if it is 3-connected to U. For example, if in a diagram Γ there exists two disjoint values of $A \subseteq T$ such that $\#E'_A = 3$ then codim $\Psi(C_\Gamma) \geq 2$.

A codimension 2 image allows to eliminate the space, and a codimension 1 image allows to give it a coefficient outside the integral lattice involved.

Now, there is also the question of finding possible other configuration spaces to complete the list. Dylan Thurston [Th] imagined to use spaces constructed in the same way as our $\mathcal{W}_m(\gamma)$ where D^2 and φ are replaced by C_θ and the map Ψ defined on it. But such spaces do not need to be introduced because they can always be replaced by a suitable combination of the $\mathcal{W}_m(\gamma)$ and $W(\gamma)$ as we have done. I do not know if one can introduce usefully other spaces.

The second aspect of the problem is to determine which faces are degenerated. One can sometimes deduce that a face is degenerated from the fact that it is Ψ-isomorphic to a face of another configuration space with a codimension ≥ 2 image.

We can generalise Lemma 3.8 to the fact that a type (d) face is degenerated if $\exists B \subseteq A \cap T,\ \#(E'_B \cap E_A) = 1$, or if A is not connected.

We can also note that the type (d) faces with $\#E'_A > 4$, or $A \not\subseteq T$ and $\#E'_A > 2$, are degenerated. This is a consequence of the argument of Subsection 9.2.

The third aspect is to make a list of all possible Ψ-isomorphisms between non-degenerated faces, to enlarge the equivalence relation between faces, and thus to reduce the number of gluing conditions, so that more maps β may satisfy them.

We now recall the gluing conditions and give possible ways of weakening them:

1) The $\mathcal{R}$STU relation (see Lemma 5.3).

2) The $\mathcal{R}$IHX relation (see Lemma 5.2), that we can replace by the following weaker $\mathcal{R}$IHX' relation: a $\mathcal{R}$IHX' relation is the sum of the $\mathcal{R}$IHX relations obtained the following way: we fix the labelled diagram Γ/e where $e \subseteq T$ is the edge involved, and we let e be replaced by each absent edge.

3) For the type (d) faces, there are many possible Ψ-isomorphisms, so we could find less constraining conditions than to impose β to be constant on isomorphism classes of diagrams. All permutations of E'_A may be used. In the case of the type (c) faces with $\#E'_A = 3$, the position of A in the diagram Γ/A may be changed. But these weaker conditions are not easy to make explicit.

4) As for the anomaly, we could quotient the configuration spaces of generalised diagrams by all Ψ-isomorphisms deduced from Lemmas 6.5 and 6.6. Then, the map β defined on ordinary diagrams can be extended to generalised diagrams in a way which satisfies the resulting gluing conditions if and only if: for all diagrams obtained by inserting a connected γ into a fixed diagram Γ at different places of a fixed component of M, the expression

$$\beta(\Gamma, \gamma) + (-1)^{\deg \gamma + 1} \beta(\Gamma, -\bar{\gamma})$$

does not depend on the place where γ is inserted.

We could also wonder if some torsion Vassiliev invariant can be found by this method, when the map β takes values in a torsion $\mathbb{Z}$-module. But a torsion invariant, if it exists, might not be found by this method. (This is suggested by the fact that the rationality results we can obtain in degree 2 by this method are not the best ones: see Subsection A.3). But we won't be interested in this problem here.

A.2 Universality of Z_n in the case of a free module

Now we are going to see that if the map β takes values in a free $\mathbb{Z}$-module or in a $\mathbb{R}$-vector space, then the corresponding $\deg \xi$ if it exists, or the corresponding integral, comes from Z_n. Thus, the only interest in choosing such other maps β is to obtain better rationality results than those of Theorem 1.11.

Proposition A.1 *If a map β from the set of 4-connected to U elements of $D_{n,k}(M)$ to some $\mathbb{R}$-vector space $\mathcal{E}$ obeys the $\mathcal{R}$STU relation, then the corresponding expression of* $\deg \Psi$ *is of the form $\mathcal{L}(Z_n(L))$ where $\mathcal{L}$ is a linear map from $\mathcal{A}_n(M)$ to $\mathcal{E}$ which only depends on β.*

Let $\mathcal{G}$ be the set of diagrams $\Gamma \in \mathcal{D}_n(M)$ which are 4-connected to U and verify $u_\Gamma \geq k-1$.

We shall first define a map ψ from $\mathcal{G}$ to $\mathcal{E}$. Then, we shall prove that ψ extends to a linear map $\mathcal{L}$ defined on $\mathcal{A}_n(M)$, and that this $\mathcal{L}$ is the solution to our problem.

Let us first define ψ from $\mathcal{G}$ to $\mathcal{E}$ by:

$$\psi(\Gamma) = |\Gamma| \sum_{\substack{\Gamma' \in D_{n,k}(M) \\ \Gamma' \sim \Gamma}} \beta(\Gamma').$$

The existence of a linear extension of ψ on $\mathcal{A}_n(M)$ will be deduced from the following facts:

1) Any $\mathcal{R}$STU relation implies the corresponding STU relation for ψ.

2) This map ψ is well-defined on chord diagrams and obeys the four-terms relation.

3) The STU relation obtained from 1) allows us to express any diagram in $\mathcal{G}$ as a linear combination of chord diagrams.

Indeed, from 2) we deduce that ψ defines a linear map from the space of degree n chord diagrams with support M modulo the four-terms relation. But we know that this space is identified with $\mathcal{A}_n(M)$ [BN1], so ψ defines a linear map $\mathcal{L}$ on $\mathcal{A}_n(M)$. This map coincides with ψ on $\mathcal{G}$, thanks to 1) and 3). Thus,

$$\sum_{\Gamma \in D_{n,k}(M)} \beta(\Gamma) I_L(\Gamma) = \sum_{\Gamma \in \mathcal{G}} \frac{I_L(\Gamma)}{|\Gamma|} \mathcal{L}([\Gamma])$$

Now, to prove the identity

$$\sum_{\Gamma \in D_{n,k}(M)} \beta(\Gamma) I_L(\Gamma) = \sum_{\Gamma \in \mathcal{D}_n(M)} \frac{I_L(\Gamma)}{|\Gamma|} \mathcal{L}([\Gamma])$$

we have to check that for any diagram $\Gamma \in \mathcal{D}_n(M)$ not in $\mathcal{G}$, we have $I_L(\Gamma) = 0$ or $\mathcal{L}([\Gamma]) = 0$. Suppose that $I_L(\Gamma) \neq 0$. Then Γ is 4-connected to U, hence $u_\Gamma < k-1$. Thus, Γ can be expressed modulo STU as a linear combination of diagrams which have $k-1$ univalent vertices and are 4-connected to U. But since $\mathcal{L}$ vanishes on them, it also vanishes on Γ.

Now let us prove the claims 1), 2) and 3).

Proof of 1): Consider a $\mathcal{R}$STU relation; let Γ be any diagram in this relation, and let A be the pair of vertices involved.

Note that this $\mathcal{R}$STU relation is defined by the labelled diagram Γ/A. We shall consider the sum of this relation with respect to the relabellings of Γ/A, in other words, with respect to the (Υ, λ), where Υ is a diagram isomorphic to Γ/A, and λ is an isomorphism from Γ/A to Υ.

This sum of relations, when applied to one of the three parts of the relation, is a sum over the set of relabellings of Γ, that is, the set of pairs (Γ', λ) where λ is an isomorphism from Γ to Γ'. This set is in one-to-one correspondence with

the product of the set of automorphisms of Γ with the set of diagrams which are isomorphic to Γ. So, this sum gives precisely the corresponding term $\psi(\Gamma)$ of the STU relation.

This proves 1). □

Proofs of 2) and 3): Since $k \leq 2n$, ψ is well-defined on the chord diagrams, and also on the diagrams with only one trivalent vertex. Then the four-terms relation is a consequence of the STU relation deduced from 1).

The proof of 3) is easy by induction: we just have to check that when we use a STU relation to express a diagram in terms of two diagrams with one more univalent vertex, then these two diagrams are 4-connected to U provided that the first one is. □

This can be generalised to the case when β takes values in any free $\mathbb{Z}$-module M, for M is canonically imbedded in the real vector space $M \otimes \mathbb{R}$.

A.3 Rationality results in degree 2

In this subsection, we shall restrict ourselves to the case when L is a knot K.

Theorem 1.11 gives the same rationality result as the one of [Le], that is, a denominator of 48. But the actual denominator is 24, according to the formula

$$Z_2 = \frac{I_K(\theta)^2}{8}[\theta]^2 + (v_2 - \frac{1}{24})\left[\bigotimes\right]$$

where $v_2 \in \mathbb{Z}$ is the degree 2 Vassiliev invariant which vanishes on the unknot O and takes the value 1 on the trefoils. This formula comes from the Vassiliev universal property of Z (Section 8), and the computation of

$$I_O\left(\bigcirc\!\!\!\!\bigcirc\right) = \frac{1}{8}$$

first made in [GMM].

This actual denominator can be obtained in terms of degree theory as Polyak and Viro did [PV]. Let us recall their remarks, and specify the general context for which this improvement in degree two is a particular case.

Keep $N = 3$, but consider the map β from $D_{2,3}$ to $\mathbb{Z}$ defined by:

$$\beta\left(\bigcirc\!\!\!\!\bigcirc\right) = 0, \quad \beta\left(\bigotimes\right) = \frac{1}{24}$$

independently of the labelling, and

$$\beta\left(\bigcirc\!\!\!\!\bigcirc\right) = 0 \text{ or } -\frac{1}{24}$$

according to the following rule: the labellings of this diagram are divided into two classes according to the parity of the number of (oriented) edges which point to the trivalent vertex. Then give the value $-\frac{1}{24}$ to a class and 0 to the other. It

is easy to see that the gluing conditions are realised for this map β. Therefore, this induces a degree

$$\deg \Psi = v_2 - \frac{1}{24}.$$

The expression of Z_2 suggests that it would be more relevant to study the denominators of $\frac{Z(K)}{Z(O)}$.

Lemma A.2 *The expression*

$$\frac{Z(K)}{Z(O)}$$

is equal to the expression defined the same way as Z, with a long knot*: instead of an imbedding of S^1 into $\mathbb{R}^3$, it will be the imbedding of $\mathbb{R}$ into $\mathbb{R}^3$ which coincides with the inclusion of a given straight line l_z of $\mathbb{R}^3$ outside some compact set.*

This is a direct consequence of our results on the extension of the configuration space integral for q-braids and tangles (Subsection 1.4). □

For this new shape of knot, we have to replace the cyclic order on the univalent vertices of the labelled diagrams, by a total order. We are going to see the following lemma:

Lemma A.3 *The same conditions as those of Subsection A.1 but in the case of a long knot, imply that Ψ admits a degree.*

Let us first define the compactification of the configuration spaces C_Γ we use here. We imbed each C_Γ into the compact space

$$\mathcal{H} \times [-\infty, +\infty]^U.$$

This compactification has the same general properties as in the case of compact knots, except that it has new faces which correspond to the $(f, f') \in \overline{C_\Gamma}$ such that $f'(U)$ meets $\pm\infty$.

We have to check that these new faces are also degenerated. We suppose (thanks to Remark 3.5) that Γ is connected. Now, the fact that $f'(U)$ meets $\pm\infty$ implies that f_V maps all univalent vertices to the straight line l_z. We conclude that this face is degenerated by means of the argument of Subsection 9.2.

Now, let us glue the anomalous faces. Note that the tangent map $\mathrm{d}K : [-\infty, +\infty] \longrightarrow S^2$ verifies $\mathrm{d}K(-\infty) = \mathrm{d}K(+\infty)$, so defines a map from S^1 to S^2, which is the boundary of some map $\varphi : D^2 \longrightarrow S^2$. It still has the property that $I_L(\theta) + 2I$ is an integer (Lemma 6.16). (The only difference is that this integer is now even, whereas it was odd in the case of an ordinary knot.) □

With this new problem, if we choose a map β which only depends on the diagram with a cyclic order obtained by closing the total order, then this gives exactly the same rationality result as the one we would obtain with this map from the diagrams with a cyclic order, for the compact knot. So, we can obtain a better rationality result for $\frac{Z(K)}{Z(O)}$ than for $Z(K)$, only when looking for maps β which do depend on the cut of the cyclic order.

In particular for $n = 2$ and $N = 3$, take the map β which only depends on the orientations of the edges, such that

$$\beta\left(\text{ }\right) = \frac{1}{6}, \qquad \beta\left(\text{ }\right) = -\frac{1}{6}$$

and which vanishes on any other diagram. The corresponding expression $\deg \Psi$ is equal to v_2.

(Actually, no better result can be obtained with another map β when $N = 3$).

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