A note on the Configuration space integral

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The aim of this article is to prove the three claims of Remark 1.12 of our previous article, math.GT/0005085 ,"the configuration space integral for links and tangles in R^3":

– The more concise result for the denominators of the integral

– The characterisation of the diagrams Γ such that $\Psi(C_\Gamma)$ has nonepty interior for a generic L.

– The cancellation of the anomaly in degree 5.

1 Recall of definitions

Let M be a compact one-dimensional manifold with boundary. Let L denote an imbedding of M into $\mathbb{R}^3$. We say that L is a *link* if we moreover have the condition that the boundary of M is empty. And it is a *knot* when $M = S^1$.

A *diagram with support* M, is the data of :

1) A graph made of a finite set $V = U \cup T$ of vertices and a set E of edges which are pairs of elements of V, such that : the elements of U are univalent (belong to precisely one edge) and the elements of T are trivalent, and every connected component of this graph meets U.

2) An isotopy class σ of injections i of U into the interior of M.

For a diagram Γ, the *degree* of Γ is the number

$$\deg \Gamma = \frac{1}{2}\#V = \#E - \#T.$$

Let $|\Gamma|$ denote the number of automorphisms of Γ.

Let $\mathcal{D}(M)$ denote the set of diagrams with support M up to isomorphism, and $\mathcal{D}_n(M)$ its subset made of the degree n diagrams, for all $n \in \mathbb{N}$.

$\mathcal{D}_1(S^1)$ has only one element, which will be denoted by θ, for it looks like the letter θ.

An *orientation* o of a diagram Γ is the data of an orientation of all its vertices, where the *orientation* of a trivalent vertex is a cyclic order on the set of the three edges it belongs to, and the *orientation* of a univalent vertex is a local orientation of M near its image by i.

Let $\mathcal{A}_n(M)$ denote the real vector space generated by the set of oriented diagrams of degree n, and quotiented by the AS, IHX and STU relations. The image of an oriented diagram Γ in the space $\mathcal{A}_n(M)$ is denoted by $[\Gamma]$.

Now let $\mathcal{A}(M)$ denote the space

$$\mathcal{A}(M) = \prod_{n \in \mathbb{N}} \mathcal{A}_n(M).$$

A *configuration* of a diagram $\Gamma \in \mathcal{D}(M)$ on L is a map $f : V \to \mathbb{R}^3$, which is injective on each edge, and such that $\exists i \in \sigma,\ f_{|U} = L \circ i$. The *configuration space* $C_\Gamma(L)$ (or simply C_Γ if L is fixed) of a diagram $\Gamma \in \mathcal{D}(M)$ on L is the set of these configurations.

C_Γ has a natural structure of a smooth manifold of dimension $\#U + 3\#T = 2\#E$. We have shown that there is a natural orientation of C_Γ depending on the data of an orientation of Γ and an orientation of all edges of Γ, such that it is reversed when we change the orientation of one vertex or one edge.

When orientations of all edges are chosen, we have the following canonical map Ψ from C_Γ to $(S^2)^E$: for a configuration $x \in C_\Gamma$ and $e = (v, v')$, $\Psi(x)_e$, is the direction of $x(v') - x(v)$.

Let Ω denote the invariant volume form on $(S^2)^E$ with total mass 1.

Now the *configuration space integral* is $Z(L) \in \mathcal{A}(M)$:

$$Z(L) = \sum_{\Gamma \in \mathcal{D}(M)} \frac{I_L(\Gamma)}{|\Gamma|}[\Gamma]$$

where

$$I_L(\Gamma) = \int_{C_\Gamma(L)} \Psi^* \Omega.$$

The sign of the product $I_L(\Gamma)[\Gamma]$ does not depend on choices of orientations. We have proven that this integral $I_L(\Gamma)$ converges, by showing that there exists a compactification of C_Γ into a manifold with corners $\overline{C_\Gamma}$, on which the canonical map Ψ extends smoothly. Then $Z(L)$ also converges because it is a finite sum in each $\mathcal{A}_n(M)$.

For each component m of M, we denote by θ_m the degree one diagram made of a chord with both vertices on m, with the same orientation. The integrals $I_L(\theta_m)$ are the Gauss integrals of the respective components of the link. During isotopies, they can all independently run over $\mathbb{R}$.

Let $J = [0, 1]$. The bilinear product on $\mathcal{A}(J)$ defined by gluing the two copies of J preserving the orientation, defines a commutative algebra structure with unit $1 = [\emptyset]$, and $\mathcal{A}(J)$ can be identified with $\mathcal{A}(S^1)$.

Similarly, for every M and every connected component m of M with an orientation (either with boundary or not), the insertion of J at any point of m preserving the orientation provides an $\mathcal{A}(J)$-module structure on $\mathcal{A}(M)$.

We have the fundemental property of the configuration space integral of links:

$$Z(L) = Z^0(L) \prod_m \exp\left(I_L(\theta_m)\alpha^{(m)}\right)$$

where m runs over the connected components of M, and :

$Z^0(L) \in \mathcal{A}(M)$ is an isotopy invariant of L,

α is a universal constant in $\mathcal{A}(J)$ called the *anomaly*,

$\alpha^{(m)}$ means α acting on $\mathcal{A}(M)$ by the component m.

For any nonempty subset A of $V = U \cup T$, we denote

$$\begin{array}{l} E_A = \{e \in E | e \subseteq A\} \\ E'_A = \{e \in E | \#(e \cap A) = 1\} \end{array}.$$

A diagram Γ is said to be 4-connected to U if for any $A \subseteq T$ such that $\#A > 1$ we have $\#E'_A \geq 4$.

We have defined in Section 4.1 the set $D_{n,k}(M)$ of labelled diagram of a fixed degree n, which is equivalent to orienting all edges and choosing an injection of their set into $E = \{e_1, \cdots, e_N\}$ where we suppose that $k = 3n - N \leq 3$ and $n \geq 2$ (so that such an injection always exists). The image of this injection is denoted by E^v (set of "visible edges"), and its complement is denoted by E^a (set of "absent edges"). Then, the map Ψ is extended to a map from $C_\Gamma \times (S^2)^{E^a}$ to $(S^2)^N$ by the identity on $(S^2)^{E^a}$.

We have proved that for integer values of the $I_L(\theta_i)$, the integral Z can be seen at each degree n to be the degree of a map, that is, Z_n is a rational linear combination of counts (with signs according to orientations) $n_\Gamma(s)$ of the preimages of any generic point s of $(S^2)^N$ by the maps Ψ defined on the configuration spaces of a certain list of diagrams Γ: the result of this combination does not depend on the choice of s. The set of these diagrams is larger than $D_{n,k}(M)$ but the combination of integrals for the diagrams not in $D_{n,k}(M)$ vanishes.

2 Recall of the statements to prove

First statement. If for all m, $I_L(\theta_m)$ is an integer, then Z_n belongs to the integral lattice generated by the elements

$$\frac{[\Gamma]}{e_\Gamma!\, 2^{e_\Gamma}}$$

where Γ runs over the diagrams such that $\Psi(C_\Gamma)$ has codimension 0.

Second statement. For a generic L, $\Psi(C_\Gamma)$ has codimension 0 if and only if Γ is 4-connected to U.

Third statement. The degree 5 part of the anomaly cancels (this uses a Maple program).

3 Denominators of the configuration space integral

We saw in our previous article that Z_n is equal to a combination of the count of preimages of any generic point of $(S^2)^N$ in all configuration spaces of generalised diagrams C_Γ with coefficients

$$\frac{(N-e_\Gamma)!}{N!\,2^{e_\Gamma}}[\Gamma].$$

The set of the generalised diagrams is an extension of the set of "normal" diagrams we recalled, but the set of coefficients given to the generalised diagrams in this combination is essentially equal to the set of coefficients of the normal diagrams, and the new configuration spaces have some common properties with the normal ones, so that the following proof, although focused on normal diagrams only, in fact automatically extends to the whole set of generalised diagrams, so that the conclusion remains.

From the above formula to the one we want to prove, the difference can be interpreted two ways: one way is to say that we can forget about the absent edges and let $N = e_\Gamma$ (we could not simply do it because $N \geq e_\Gamma$ needs to be fixed whereas e_Γ varies); the other way is to see that there is a factor $\frac{N!}{e_\Gamma!(N-e_\Gamma)!}$ between them.

The idea of the proof is to find some well-chosen generic element of $(S^2)^N$ and prove the the rationality result at it, by giving an equivalence relation over the set $D_{n,k}(M)$ of diagrams such that each class is made up of $\frac{N!}{e_\Gamma!(N-e_\Gamma)!}$ isomorphic diagrams which give the same contribution.

We suppose that we have chosen a semi-algebraic representant of the link in its (C^1, C^2 by parts) isotopy class.

For any diagram $\Gamma \in D_{n,k}(M)$ together with a vertex orientation, let c_Γ be the map from $(\mathbb{R}^2)^N$ to $\mathbb{Z}$ defined by: $\forall\, r = (a_1, b_1, a_2, b_2, \cdots, a_N, b_N)$, $c_\Gamma(r) = n_\Gamma(s)$ where $s \in (S^2)^N$ is the family of unit vectors with same directions and senses as $(1, a_i, b_i)_{1\leq i\leq N}$.

The proof starts from the following remark:

Remark 3.1 For any labelled generalised diagram Γ, the map c_Γ is semi-algebraic and factorises through the canonical projection to $(\mathbb{R}^2)^{E^v}$, where E^v is the set of visible edges of Γ.

We recall that for any semi-algebraic map with p variables and values in $\mathbb{Z}$, its limit when one variable tends to $+\infty$ is well-defined and it is a semi-algebraic map with $p-1$ variables.

Let $c = (c_\Gamma)_{\Gamma\in D_{n,k}(M)}$, assimilated to a semi-algebraic map from $(\mathbb{R}^2)^N$ to $\mathbb{Z}^{D_{n,k}(M)}$.

Consider now the element of $\mathbb{Z}^{D_{n,k}(M)}$ defined by the expression

$$\begin{aligned} c^{lim} &= (c_\Gamma^{lim})_{\Gamma\in D_{n,k}(M)} \\ &= \lim_{a_1\to+\infty}(\lim_{a_2\to+\infty}(\cdots(\lim_{a_N\to+\infty}(\lim_{b_1\to+\infty}(\lim_{b_2\to+\infty}(\cdots(\lim_{b_N\to+\infty} c)\cdots)))) \end{aligned}$$

We can note that c_Γ^{lim} only depends on the order between the visible edges in the set $\{1, \cdots N\}$ and not on the positions of the absent edges between them (provided that the vertex orientations are the same), because the operation of passing to the limit has no effect for absent edges.

Thus, the classes of diagrams with the same order between visible edges and which only differ by the positions of absent edges between them, obey the required properties.

This proves the first statement. $\square$

4 Linearisation of Ψ and signs of contributions of the preimages of a point

For practical reasons, to count the preimages of a point of $(S^2)^N$, we shall put together the contributions of the diagrams with different orientations of the edges. Thus, we shall use the quotient of the set $D_{n,k}(M)$ of diagrams by the changes of edge orientation, and then choose an arbitrary representant in each class of this partition. The different orders between edges still give different diagrams in this new classification. This way, one count of preimages of a point of $(S^2)^N$ includes all configurations where the edges, taken in their fixed order, have the same respective required directions but not necessarily the required senses. The sign of the contribution of a preimage will, of course, depend on these senses.

Here we suppose that there are no absent edges, so N is the number of edges of Γ. So, let us choose a point $s = (s_e)_{e \in E} \in (S^2)^N$, and let

$$\mathcal{E}_s = \prod_{e \in E} \mathbb{R}^3 / \mathbb{R} s_e .$$

We said that a configuration space C_Γ is a set of maps f from the set V of vertices to $\mathbb{R}^3$ with specified conditions. Now, let Ψ_s be the linear map from $(\mathbb{R}^3)^V$ to $\mathcal{E}_s$ defined by: for each edge $e = (v, v') \in E \subseteq V^2$, and each $f \in (\mathbb{R}^3)^V$, $\Psi_s(f)_e \in \mathbb{R}^3/\mathbb{R}s_e$ is the image of $f(v') - f(v)$ by the canonical linear projection.

For the case of a link L, Ψ_s is defined on $C_\Gamma(L)$ which is a submanifold of $(\mathbb{R}^3)^V$ with the same dimension as $\mathcal{E}_s$. Then, the set of preimages of 0 by Ψ_s in $C_\Gamma(L)$ corresponds bijectively to the disjoint union of sets of preimages of s by Ψ in all the $C_{\Gamma'}$ where Γ' differs from Γ by a change of edge orientation.

Now, we have to define the sign of each preimage in order to compute the global contribution of the diagrams Γ' to the "degree of Ψ_n", and thus to Z_n. All these diagrams Γ' have the same image $[\Gamma'] = [\Gamma]$ in $\mathcal{A}_n$ provided that they only differ by edge orientation and keep the same vertex orientation.

The sign of a preimage is the sign of the determinant of the tangent map from a tangent space of $C_\Gamma(L)$ into $\mathcal{E}_s$, for proper orientations of these spaces. If we fixed their orientations, this sign of deterninant would not vary when we change the edge orientation of Γ, because when we reverse one edge, the orientation of $f(v') - f(v)$ changes but it corresponds to two dimensions of $\mathcal{E}_s$.

Then, we note that the orientation of $\mathcal{E}_s$ is determined by the orientations of vectors s_e which we see as fixed, whereas the orientation of the tangent space to C_Γ changes whenever we change one edge orientation of Γ. And this edge orientation of Γ' is defined for one given preimage of 0 by Ψ_s in $C_\Gamma(L)$ by the fact that all $f(v') - f(v)$, which are colinear with the respective s_e, must also respect their senses. So, each change of orientation of an $f(v') - f(v)$ compared to s_e induces a sign change of the contribution. We conclude that, from a conventional arbitrary choice of edge orientation:

$$\mathrm{sgn}(f \in C_\Gamma(L) \cap (\ker \Psi_s)) = \det \Psi_{s|T_f C_\Gamma(L)} \times \prod_{e=(v,v') \in E} (\mathrm{sgn}(\lambda) | f(v') - f(v) = \lambda s_e).$$

5 Proof of the second statement

We recall that one sense of the implication has already been proven, and the converse can be reformulated in the form: "If Γ is 4-connected to U then there exists a link L such that $\Psi(C_\Gamma)$ has codimension 0".

The condition "$\Psi(C_\Gamma)$ has codimension 0" can be reformulated in the form "There exists a configuration $x \in C_\Gamma$ such that the linear map $d\Psi(x)$ is bijective", in other words, denoting $s = \Psi(x)$, Ψ_s is bijective from the tangent space of C_Γ at x to $\mathcal{E}_s$.

And the tangent space of C_Γ at x is the topological closure $\overline{C_\Gamma}(L')$ of the space $C_\Gamma(L')$ where L' is the family of the tangent straight lines to the link L at the positions of univalent vertices in the configuration x.

Now, let us put a question which seems completely different from the above one (because if $\Psi(C_\Gamma)$ has not codimension 0, then the elements $s = \Psi(x)$ of the image of Ψ are not generic):

For a generic $s \in (S^2)^N$ and a generic family L' of affin lines, is the (linear) map Ψ_s bijective from the affine space $\overline{C_\Gamma}(L')$ to $\mathcal{E}_s$?

First case: Ψ_s is bijective from $\overline{C_\Gamma}(L')$ to $\mathcal{E}_s$ for generic s and L'.

Then 0 has one preimage x in $\overline{C_\Gamma}(L')$. The question is to know whether it is in $C_\Gamma(L')$ or not. If it is, then we can find a link which is tangent to every line of L' at the position of every univalent vertex of x, so we are done.

If x is not in $C_\Gamma(L')$, then it is the configuration space of the quotient of Γ by a certain partition of its vertices. Note that in this partition, no two univalent vertices can go together because L' has been supposed to be generic, so all univalent vertices go to different and disjoint straight lines.

Now, fix L' and let s vary. Then x contains the information on the s_e for the e which are not collapsed in this quotient. Therefore the configuration space of the quotient diagram must have at least the dimension of the space of variation of these s_e. So, the quotient diagram must have a collapsed vertex which may run over $\mathbb{R}^3$ and is no more than trivalent, so that Γ is not 4-connected to U.

Second case: Ψ_s is not bijective from $\overline{C_\Gamma}(L')$ to $\mathcal{E}_s$ for generic s and L'.

Then let us consider a nonzero element x of the linear part of this map. So it is a configuration with support lines passing through the origin, with generic directions.

As above, we consider a quotient diagram. This time, several univalent vertices may collapse into one at the origin but the configuration space passes through the quotient by a scalar multiplication, thus the generic s cannot be reached unless the quotient diagram has a bivalent vertex resulting from a collapse of trivalent vertices, in which case Γ is even not 3-connected to U.

This ends the proof of the second statement.

6 Anomaly as a degree for configurations with support two vertical lines

Let $\mathcal{D}^c(2)$ be the set of the nonempty connected diagrams in $\mathcal{D}(\mathbb{R}\sqcup\mathbb{R})$ for which the univalent vertices do not all go to the same component.

Let $\mathcal{P}$ be the closed subspace of $\mathcal{A}(\mathbb{R}\sqcup\mathbb{R})$ generated by the elements of $\mathcal{D}^c(2)$ at each degree.

Let $\tilde{\alpha}$ be defined the same way as the configuration space integral Z except that the set of diagrams is replaced by $\mathcal{D}^c(2)$ and the link L is replaced by two vertical lines, so the dimension of the configuration spaces decreases by one because of the vertical translations which preserve Ψ, and increases by one again because we let the relative positions of the vertical lines vary by one whole turn.

We can deduce from Subsection 9.5 the relation

$$-2\alpha = \tilde{\alpha}.1_{\cap}$$

This uses the fact that the volume form of S^2 involved in the integral is invariant by rotations around the vertical axis.

This will be the formula used to compute the anomaly. But to compute the integral defining $\tilde{\alpha}$, we would like it to correspond to a degree of a map, which means that the number of preimages of an element $s \in (S^2)^N$ with wheights in $\mathcal{A}_n(\mathbb{R} \sqcup \mathbb{R})$ would not depend on s. This is not exactly true, but becomes true once it is mapped by $1_{\cap}$ into $\mathcal{A}_n$, because this the degree n part of an algebra morphism of $\mathcal{A}(\mathbb{R} \sqcup \mathbb{R})$ into a commutative algebra. All gluigs can be defined between diagrams the same way as we did for configuration spaces of links, except that some $RSTU$ relations may now involve disconnected diagrams. But the couples (generic points of the face, disconnected diagram of the relation) can be grouped by pairs in the way uniquely defined by the following property. In each pair, the disconnected diagram is the same, and the configurations differ by the fact that the two components are vertically translated one with respect to the other. Here the commutativity of the algebra after the map $1_{\cap}$ is used in the case of non-overlapping components.

7 Eliminating some diagrams

The previous section allows us to compute the degree n part of the anomaly ($n > 2$) as a linear combination of the numbers (with signs) of preimages by Ψ of any generic point of $(S^2)^N$ in the configuration spaces of diagrams with support two vertical lines.

Now, we want to know for which diagrams there can be preimages, or in other words, which of these configuration spaces have a codimension 0 image by Ψ.

Proposition 7.2 *The configuration space of a diagram Γ with support two vertical lines has codimension 0 image by Ψ if an only if Γ is 4-connected to U and each $A \subseteq V$ such that $\#A \geq 2$ and $\#E'_A = 1$ contains univalent vertices on both lines.*

Proof. For a given diagram Γ, let $\mathcal{C}_\Gamma$ be the space of "configurations" where all trivalent vertices run over $\mathbb{R}^3$ and all univalent vertices run over their respective vertical line, except that the position of one fixed univalent vertex is, by convention, fixed at the origin. It is a $(2N+1)$-dimensional vector space (where N is the number of edges). The configuration spaces for diagrams with different orders of the univalent vertices in each vertical line are disjoint open subsets of it, and their closures cover it. So they will be treated as a same case (their images have equal codimensions).

If there exists $A \subseteq V$ such that $\#A \geq 2$ and $\#E'_A = 1$ whose univalent vertices are contained in one line, the proof that the configuration space has an empty interior image by Ψ is similar to the case of a diagram which is not 4-connected to U: the subconfiguration of the vertices in A modulo dilations and translations has a strictly lower degree of freedom than twice the number of edges in A, whose directions are determined by them.

Now we have to prove the converse.

In a similar way as in the proof of the second statement, we take a generic s and define a linear map Ψ_s from $\mathcal{C}_\Gamma$ to $\mathcal{E}_s$. Since the dimension of $\mathcal{C}_\Gamma$ is one more than the dimension of $\mathcal{E}_s$, the kernel of this map is at least 1-dimensional.

Let x be a nonzero element of $\operatorname{Ker} \Psi_s$. The question is to know whether x is an acceptable configuration if s is generic, where an acceptable configuration is a configuration where both vertical lines are distinct, all univalent vertices are distinct and no edge is collapsed.

If x is not acceptable, consider the space of the same kind of collapsed configurations as x (modulo dilations and translations), and what remains of the map Ψ from it (giving the directions of the noncollapsed edges). Since s is generic, this collapsed configurations space must have at least the dimension of the space of arrival. Thus, we can see that the two vertical lines must remain distinct, any subdiagram A collapsed onto one point not following one of them must have at most three exterior edges ($\#E'_A \leq 3$) and any subdiagram A collapsed onto one point following one of them must have at most one exterior edge.

Therefore, non-acceptable elements of Ψ_s can exist for a generic s only if Γ is not 4-connected to U or if there exists $A \subseteq V$ such that $\#A \geq 2$ and $\#E'_A = 1$ whose univalent vertices are contained in one line. This ends the proof of the proposition.

Corollary 7.3 *Only the diagrams with at least two univalent vertices on each line can contribute.*

If u was the only univalent vertex on one line, the set $A = U - \{u\}$ would have the above property because the diagram has a degree $n > 2$.

Proposition 7.4 *(Lescop) If a diagram has not at least 3 univalent vertices on each line, the contributions of diagrams in the same isomorphism class (with different orders between edges) cancel each other.*

Consider a diagram with only two univalent vertices u and u' on one line and let $e = \{u, v\}$, $e' = \{u', v'\}$ be their edges. Let f be a configuration of this diagram. Suppose by convention that the line of u and u' is not the fixed line. Now, let f' be another configuration defined by:

$$\begin{aligned} f'(u) &= f(v) + f(v') - f(u') \\ f'(u') &= f(v) + f(v') - f(u) \\ f &= f' \text{ elsewhere} \end{aligned}$$

Then we have

$$\begin{aligned} f'(u) - f'(v) &= f(v') - f(u') \\ f'(u') - f'(v') &= f(v) - f(u) \\ f'(u') - f'(u) &= f(u') - f(u) \end{aligned}$$

Thus, $s = \Psi(f)$ and $s' = \Psi(f')$ only differ on e and e' where $s'_e = -s_{e'}$ and $s'_{e'} = -s_e$.

Now, let us prove that the contributions of these two configurations have opposite signs.

The orientation of $(S^2)^N$ is unchanged: this transformation only exchanges s_e and $-s_{e'}$ which are 2-dimensional quantities.

The orientation of the configuration space is reversed because it exchanges the dimension of variation of u with the one of u' (and changes signs of both, which has no effect).

This ends the proof of the proposition.

Note that the cancellation of the degree 3 part of the anomly can be seen as a corollary of this result.

Symmetries

The symmetries of the anomaly we saw in Lemmas 6.5 and 6.6 can be found again in this representation.

The first one, the S^1-symmetry, appears in the form of the exchange of the two lines, both in the diagram and geometrically, whereas the configuration remains fixed.

The second one, which multiplies the contribution of a configuration in the expression of the anomaly by $(-1)^{n+1}$, is defined by the central symmetry of the configuration together with the exchange of the lines by central symmetry in the diagram, whereas these lines remain geometrically unchanged.

The proof of the second symmetry is similar to the one done for Lemma 6.6. Here, the residual -1 sign comes from the fact that the orientation of the configuration space was obtained by quotienting by the group of vertical translations whose orientation stays fixed. In other words, there is one arbitrary vertex whose vertical coordinate cannot follow the symmetry, because it is replaced by the angular coordinate of the geometrical position of the vertical lines which is unchanged.

8 Computation method

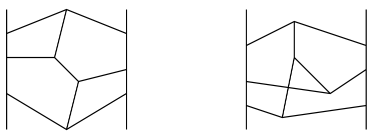
The computation of the degree n part of the anomaly (n odd) can be made by the following steps. We shall state the general rules and apply them to the case $n = 5$. This computation for $n = 5$ has been made and gave the result 0, but the file of the data has been lost. However, we give here such details of the method that repeating this computation would be very easy.

First step: list the diagrams

Make a list of all isomorphism classes of (uni-trivalent, 4-connected) degree n graphs with at least 6 univalent vertices but with no structure on them (like Chineese characters). Then, make a list up to isomorphism of the possible partitions of the univalent vertices into two classes of at least 3 elements such that (according to Proposition 7.2) whenever the diagram is cut at one edge into two parts, each part has univalent vertices in both classes (such partitions always exist). So, it says on which line every univalent vertex will be, but not the order between them on each line.

These objects (a diagram with such a partition of univalent vertices) will be the "diagrams" we shall work with.

For $n = 5$ we find only two possibilities:



So, here the number N of edges will always be 9.

Second step: label its edges and vertices

First, choose an arbitrary orientation of all trivalent vertices.

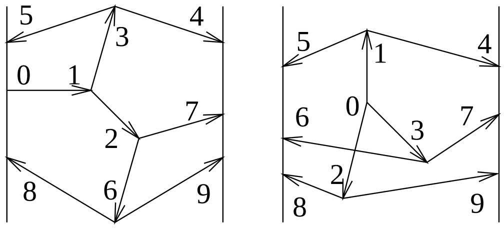
By convention, the orientations of the univalent vertices will be given by their altitudes, so all upwards. At the end, to give the class of the diagram we shall have to remember that through the map $1_\cap$ we change this convention, all univalent vertices getting the same orientation on the line or the circle, therefore changing the sign if the number of univalent vertices on one given line was odd. For $n = 5$ this number is always 3, thus reversing all signs exactly.

Then, list and orient arbitrarily all edges. In the case of a tree-connected diagram which will be our interest here, there is a "good way" of doing it. Start with an arbitrary vertex, then take and orient one edge from it. So, two vertices have been reached, at least one of which is trivalent and is thus connected to two other edges.

Now, repeat the following procedure until all the diagram has been covered:

Choose a trivalent vertex that has been reached but still not used. Then take and orient the two remaining edges form it, in the order defined by the orientation of this vertex.

For $n = 5$ we can obtain the following:



This gives a basis for the $2N+1$-dimensional space $\mathcal{C}_\Gamma$, by using the method to orient the configuration space, where all third (vertical) coordinates are chosen to correspond to the ends of edges. The first vertex, where no edge ends, will be the vertex whose third coordinate is replaced by the angular coordinate of the vertical lines. But we introduce the radial coordinate before all this. Through a rotation, these radial and angular coordinates are respectively replaced by the x and y cartesian coordinates of the second line with respect to the first. This gives for the first diagram the following order of the coordinates in $\mathcal{C}_\Gamma$, numbered from 0 to $2N$, determined by the drawn realisations of the above procedure:

$$(x, y, z_1, x_1, z_2, y_1, z_3, x_3, z_4, y_3, z_5, x_2, z_6, y_2, z_7, x_6, z_8, y_6, z_9)$$

for the first diagram, and

$$(x, y, z_1, x_0, z_2, y_0, z_3, x_1, z_4, y_1, z_5, x_3, z_6, y_3, z_7, x_2, z_8, y_2, z_9)$$

for the second.

Third step: Formally compute Ker Ψ_s for a variable s.

As we did in the proof of the first statement, we shall represent a generic s by a family of variables $(a_1, b_1, a_2, b_2, \cdots, a_N, b_N) \in (\mathbb{R}^2)^N$.

The linear map Ψ_s from $\mathcal{C}_\Gamma$ to $(\mathbb{R}^2)^N$ is defined by: for each edge $e_i = (u, v)$ (where i runs from 1 to N), and each $f = (x, y, \cdots, z_N) \in \mathcal{C}_\Gamma$,

$$(\Psi_s(f))_i = (y_v - y_u - a_i(x_v - x_u), z_v - z_u - b_i(x_v - x_u)).$$

This directly gives the matrix of Ψ_s. So, it is a matrix P with $2N$ rows numbered from 1 to $2N$ and $2N + 1$ columns numbered from 0 to $2N$. It is of rank $2N$, with plenty of variable symbols inside.

We deduce from the symmetry properties of the integral, that the contributions of the two opposite possible values of the generator of the kernel, once mapped by $1_\cap$ into $\mathcal{A}(S^1)$, are S^1-symmetric of each other.

A standard result of matrix theory gives the following formula for a generator g of the kernel: its i-th coordinate is $(-1)^i$ times the determinant of the matrix P whose i-th column has been deleted. But not only this gives the direction of the kernel, this also orients it the good way so that the sign of the contribution can be computed. Indeed, let us recall the defining property of g: for any linear form l on $\mathcal{C}_\Gamma$, $l(g)$ equals the determinant of the matrix obtained by putting on top of P the row equal to the row-matrix of l. Now, if we choose l to be the differential of $\frac{1}{2}(x^2+y^2)$ at g (measuring the growing of distance between the vertical lines), then $l(g)=x^2+y^2>0$ (for a generic s, as we showed). Hence, the matrix obtained by putting l on top of P has a positive determinant. This l at the top is a measure of the dilation of the configuration around g. Its role as 0-th element in the basis of $\mathcal{C}^*_\Gamma$ defined by P so corresponds for the orientation to the role we gave to x as 0-th coordinate in $\mathcal{C}_\Gamma$.

So, the sign of the contribution of g is just the product of the geometrical orientation of the edges, which is what we have to compute. For each edge $e_i=(u,v)$, we need to find the sign of the first coordinate of its vector, x_v-x_u, because this forms a positively oriented dual base of $\mathbb{R}^3$ with the linear forms $(-a,1,0)$ and $(0,-b,1)$ that we choosed as dual base of $\mathbb{R}^3/(1,a,b)$ to represent Ψ_s.

Therefore, we must first compute with the deteminants the x_i's coordinates of g cited above. In our case $n=5$, there are only five such determinants to compute for each diagram. Moreover, using their tree-connectedness and the particular procedure we used to label the coordinates, all vertical coordinates (the rows and colums with nonzero even label) immediately disappear in this computation from the last to the first, as for the determinant of an inferior triangular matrix with all coefficients 1 in the diagonal.

You can ask: then, what could we do if the diagram was not tree-connected ? The solution would be to start by manipulating the raws of vertical coordinates until they disappear all but a number of raws representing the loops in the graph, which form a base of its first homology.

In our case, this simplifies the matrix in the following way: for the first diagram, we get the following matrix with colums representing the coordinates $(x,y,x_1,y_1,x_3,y_3,x_2,y_2,x_6,y_6)$:

$$\begin{pmatrix}
0 & 0 & -a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_2 & -1 & 0 & 0 & -a_2 & 1 & 0 & 0\\
0 & 0 & a_3 & -1 & -a_3 & 1 & 0 & 0 & 0 & 0\\
-a_4 & 1 & 0 & 0 & a_4 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a_5 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_6 & -1 & -a_6 & 1\\
-a_7 & 1 & 0 & 0 & 0 & 0 & a_7 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & -1\\
-a_9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & -1
\end{pmatrix}$$

and for the second, with coordinates $(x, y, x_0, y_0, x_1, y_1, x_3, y_3, x_2, y_2)$,

$$\begin{pmatrix} 0 & 0 & a_1 & -1 & -a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & -1 & 0 & 0 & 0 & 0 & -a_2 & 1 \\ 0 & 0 & a_3 & -1 & 0 & 0 & -a_3 & 1 & 0 & 0 \\ -a_4 & 1 & 0 & 0 & a_4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 & -1 & 0 & 0 \\ -a_7 & 1 & 0 & 0 & 0 & 0 & a_7 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & -1 \\ -a_9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & -1 \end{pmatrix}.$$

So, the expressions of x and the x_i coordinates can be computed from this new matrix by deleting the corresponding column and taking the determinant (with no other sign because the expression of the determinant defining the component x is unchanged from the previous matrix to the new one and thus stays without sign; the other x_i components also stay without sign because they have the same parity as x in the new matrix and correspond, together with x, to a generator of the kernel of the new matrix whose linear algebra construction still apply).

Then, the first coordinates of the edges can be deduced by substractions: for the first diagram, $X_1 = x_1$, $X_2 = x_2 - x_1$, $X_3 = x_3 - x_1$, $X_4 = x - x_3$, $X_5 = -x_3$, $X_6 = x_6 - x_2$, $X_7 = x - x_2$, $X_8 = -x_6$, $X_9 = x - x_6$.

They are big polynomials in variables a_i which will be used two ways.

Their first use is that the product of their signs gives the sign of the contribution.

Their second use is that the orders of the univalent vertices on each line are simple functions of them: we have for the first diagram the following expressions which can be immediately read from it

$$\begin{aligned} -z_8 &= -b_1X_1 - b_2X_2 - b_6X_6 - b_8X_8 \\ z_5 &= b_1X_1 + b_3X_3 + b_5X_5 \\ z_5 - z_8 &= b_3X_3 + b_5X_5 - b_2X_2 - b_6X_6 - b_8X_8 \\ z_7 - z_9 &= b_7X_7 - b_6X_6 - b_9X_9 \\ z_4 - z_7 &= b_3X_3 + b_4X_4 - b_2X_2 - b_7X_7 \\ z_4 - z_9 &= b_3X_3 + b_4X_4 - b_2X_2 - b_6X_6 - b_9X_9 \end{aligned}$$

and some similar expressions for the second diagram.

Unfortunately, such tricks cannot be directly generalised to the case of a non tree-connected diagram, because the coefficients b_i do occur in the expressions of the X_i.

Forth step: Run all permutations of the edges to collect the contributions

This is the step for which computer help is the most necessary.

What we want to compute is, for all permutations p of $\{1, \cdots 9\}$, the signs of all nine polynomials of the edges $X_1, \ldots, X_9$, and all six polynomials of the heights $z_5, \ldots, z_4 - z_9$, at the super-limit value of the variables

$$\lim_{a_{p(9)}\to+\infty}(\lim_{a_{p(8)}\to+\infty}(\cdots(\lim_{a_{p(1)}\to+\infty}(\lim_{b_{p(9)}\to+\infty}(\lim_{b_{p(8)}\to+\infty}(\cdots(\lim_{b_{p(1)}\to+\infty}\bullet)\cdots)))).$$

Remark that this limit operation for the a_i is the same as for the b_i, and we do not have to repeat it: the X_i only depend on the the family (a_j), and the sign of each height difference only depends on the sign of one of the X_i, the one with the value of i corresponding to the first coefficient b_i in its expression which tends to infinity.

So, a trick that can be used is to put in the computer the data

$$\begin{aligned}
X_{10} &= -1a_1 - 2a_2 - 6a_6 - 8a_8\\
X_{11} &= 1a_1 + 3a_3 + 5a_5\\
X_{12} &= 3a_3 + 5a_5 - 2a_2 - 6a_6 - 8a_8\\
X_{13} &= 7a_7 - 6a_6 - 9a_9\\
X_{14} &= 3a_3 + 4a_4 - 2a_2 - 7a_7\\
X_{15} &= 3a_3 + 4a_4 - 2a_2 - 6a_6 - 9a_9
\end{aligned}$$

and run the following program with the polynoms X_i where i runs from 1 to 15. Remark that any variable in each polynomial occurs with at most degree 1.

Define a table of polynomials with two indexes $X(i,j)$ where $i \in [1,15]$ (the set of integers from 1 to 15) and $j \in [0,9]$, and an array of integers $P(j)$ where $j \in [1,9]$, which will denote the permutation p

For all i, let $X(i,0)$ be the polynomial X_i defined above. Open also a table $T(k,l)$ of integers with two entries $k,l \in [1,6]$, where k represents the order between the heights on the first line z_5, 0, z_8 and where l represents the order between the heights on the second line z_9, z_7, z_4, according to the bijection:

$k=$	1	$z_8<0<z_5$	$l=$	1	$z_9<z_7<z_4$
	2	$0<z_8<z_5$		2	$z_7<z_9<z_4$
	3	$z_8<z_5<0$		3	$z_9<z_4<z_7$
	4	$0<z_5<z_8$		4	$z_7<z_4<z_9$
	5	$z_5<z_8<0$		5	$z_4<z_9<z_7$
	6	$z_5<0<z_8$		6	$z_4<z_7<z_9$

(Here we shall consider the particular case of the first diagram. The case of the second diagram is similar.) This table will receive the counts of the contributions. So all its values start with 0.

Start with $j=1$.

Define the recursive procedure $Comp(j)$ to be made of the following operations, depending on j from 1 to 9.

For every value of $P(j)$ different from all $P(k)$ for all $1 \le k < j$, do the following two steps:

1) For all $i \in [1,15]$, let $X(i,j)$ be the polynom

$$\frac{\partial}{\partial a_{P(j)}} X(i,j-1)$$

if it is not zero (this is the same as the coefficient of degree 1 in $X(i,j-1)$ seen as a polynomial of degree 1 with variable $a_{P(j)}$), or else the polynom $X(i,j-1)$ itself. (This operation means the computation of the sign of the limit when $a_{P(j)}$ tends to infinity, depending on the remaining variables.)

2) If $j<9$ then call the procedure $Comp(j+1)$, else compute the contribution as follows. Define the signs for $m=1$ to 6

$$s(m)=\mathrm{sgn}(X(m+9,9))\times\mathrm{sgn}(X(|X(m+9,9)|,9))$$

Finally, add $\prod_{1\leq i\leq 9}\mathrm{sgn}(X(i,9))$ to the element

$$T(s(2)+s(3)+\frac{1}{2}(s(1)+7),s(5)+s(6)+\frac{1}{2}(s(4)+7)).$$

The procedure $Comp(j)$ is finished. Once called, the table of contributions $T(k,l)$ is completed. Then, we just have to interpret its results.

Fifth step: Put the contributions into the space of diagrams to obtain the result

Each of the (two) diagrams (with only a partition of univalent vertices in two classes) we started with, gives a table of contributions.

Now the only difficulty is to take care of the orientations and of the set of automorphisms of the diagrams by which the results must be divided.

If the forth step was done, as described, by listing *all* permutations of the edges, then we must do the following:

1) Take one table of contributions. Each element of it (corresponding to an isotopy class of injections of the univalent vertices into $\mathbb{R}\sqcup\mathbb{R}$) contributes with wheight $[\Gamma]+[\bar{\Gamma}]$ where Γ is the diagram with support S^1 obtained by closing $\mathbb{R}\sqcup\mathbb{R}$ into a circle, and $\bar{\Gamma}$ is its S^1-symmetric image. Take care of the change of orientation of the univalent vertices on one line (anyway, all diagrams in the table have the same number of univalent vertices on each line, so all signs are affected the same way).

For the case $n=5$ the computation ends here because for each of both diagrams, the contributions already cancel via the IHX and STU relations.

2) Divide the result by the number of automorphisms of the diagram we started with (with only a chosen partition of the univalent vertices in an unordered pair of sets). For $n=5$, if the computation did not already stop, these numbers of automorphims would respectively be 2 and 12.

3) Finally, sum up the results with respect to the list of diagrams with partitions of univalent vertices. The result is equal to $-\alpha=\frac{1}{2}\tilde{\alpha}.1_{\cap}$ because we quotiented the set of diagrams by the exchange of the two lines as an automorphism, using the symmetry property.

If we wanted to compute higher degrees, the big number of operations in the forth step may be a problem, so it can be useful to avoid the repetition of isomorphic configurations. To do this, in the procedure $Comp(j)$, instead of letting $P(j)$ run over all the complement of the set K of $P(k)$ for $1\leq k<j$,

only take a system of representants of its orbits by the subgroup of automorphisms which fix all elements of K. This idea would be hard to implement but theoretically possible. Then, no division by a number of automorphism would be needed.